Lagrange interpolation of bandlimited functions on slowly increasing sequences

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***Abstract*—Let $\Lambda = \{z_{n,k} : 1 \leq k \leq n,\ n \in \mathbb{N}\}$ be a triangular array of distinct complex numbers and let f be an entire function. Suppose $L_{n-1}f$ is the unique polynomial of degree at most $n-1$ which interpolates f at $z_{n,k}$ for $k \in \{1, \cdots, n\}$, i.e., $L_n f(z_{n,k}) = f(z_{n,k})$. In this note, we show that $L_n f$ converges to f uniformly on compact subsets of the complex plane provided Λ is bounded. We next consider the case when $z_{n,k} = z_k$ where $\{|z_k|\}_{k\in\mathbb{N}}$ is a slowly increasing unbounded sequence in the sense that for some $\alpha \in]0,1[$, $(k-1)^\alpha \leq |z_k| \leq k^\alpha$ for each $k \in \mathbb{N}$. If f is bandlimited, we prove as well that $L_n f$ converges uniformly (and rapidly) to f on compact subsets of the complex plane. The rate of convergence that we obtain is optimal to some extent.**

Keywords: Lagrange interpolation; signal recovery; sampling; bandlimited signals; entire functions; Jensen's formula

I. Introduction

Fix $\Omega > 0$ and let $PW(\Omega)$ denote the Paley-Wiener class, that is, the set of functions $f \in L^2(\mathbb{R})$ whose Fourier transform

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-i\omega t}\, dt$$

is supported on $[-\Omega, \Omega]$. Suppose $0 < \delta\Omega < \pi$ and ϕ is a function from the Schwartz class with $\widehat{\phi}$ supported on $[-\pi\delta^{-1}, \pi\delta^{-1}]$ and such that $\widehat{\phi} = (2\pi)^{-1/2}$ on $[-\Omega, \Omega]$. Then any f in $PW(\Omega)$ can be recovered from its values $\{f(k\delta) : k \in \mathbb{Z}\}$ by the formula $f(x) = \delta \sum_{k\in\mathbb{Z}} f(k\delta)\phi(x - k\delta)$ [**?**].

More generally, let $\{x_k\}_{k=-\infty}^{\infty}$ be a bi-infinite increasing sequence in $\mathbb{R}$, such that $\lim_{k\to\pm\infty} x_k = \pm\infty$ and $\Delta = \sup_{k\in\mathbb{Z}}(x_k - x_{k-1}) < \infty$. If $f \in PW(\Omega)$ with $\Delta\Omega < \pi$, it is well known that we may recover f from the samples $\{f(x_k)\}_{k=-\infty}^{\infty}$. In fact, in [**?**] and [**?**], the authors provide algorithms for signal reconstruction.

We pose the following question: if $\{t_k\}_{k=1}^{\infty}$ is a sequence in $]0, +\infty[$ slowly increasing to infinity, is it possible to recover $f \in PW(\Omega)$ from its samples $\{f(t_k)\}_{k=1}^{\infty}$? In this note, we prove that if for some $\alpha \in]0,1[$, $(k-1)^\alpha \leq |z_k| \leq k^\alpha$ for all $k \in \mathbb{N}$, then the Lagrange polynomials $L_{n-1}f$ interpolating f at the points $z_1, \cdots, z_n$ converge uniformly to f on compact subsets of the complex plane. The rate of convergence that we obtain is optimal to some extent.

This result answers a special case of a more general problem. Let $\{z_k\}_{k=1}^{\infty}$ be a sequence of complex numbers satisfying

$$\lim_{r\to+\infty} r^{-1} |\{k \in \mathbb{N} : |z_k| \leq r\}| = +\infty. \tag{I.1}$$

Then, any f in $PW(\Omega)$ is completely determined by its samples $\{f(z_k)\}_{k=1}^{\infty}$ (Proposition **??** below). Thus, an interesting task is to reconstruct f from $\{f(z_k)\}_{k=1}^{\infty}$. When $\{z_k\}_{k=1}^{+\infty}$ is bounded, (**??**) is satisfied and we show in Theorem **??** that Lagrange interpolation accomplishes the reconstruction. Moreover, if $(k-1)^\alpha \leq |z_k| \leq k^\alpha$ with $0 < \alpha < 1$, the condition in (**??**) is also satisfied and we also show that Lagrange interpolation does the job in Theorem **??**.

One of the most beautiful results on Lagrange interpolation goes back to the 1930's by J.L. Walsh. Suppose f is analytic on the disk $\{z \in \mathbb{C} : |z| < \rho\}$ for some $\rho > 1$. For a positive integer n, let $L_{n-1}f$ be the unique polynomial of degree at most $n-1$ which interpolates f at the zeros of $z^n - 1$. Walsh proved that

$$\lim_{n\to\infty} \left(L_{n-1}f(z) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} z^k \right) = 0$$

uniformly for $|z| \leq r$ whenever $0 < r < \rho^2$. This work is also inspired by results of Totik in [**?**] on the recovery of H^p functions on the open unit disk $\mathbb{D} =$

$\{z \in \mathbb{C} : |z| < 1\}$. Given a sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ with $\sum_{n=1}^{\infty}(1-|z_n|) = +\infty$, there exist polynomials $p_{n,j}$ such that $\sum_{j=1}^{n} f(z_j)p_{n,j}$ converges to f, for any $f \in H^p$.

We refer the reader to the book [**?**] for an excellent survey on complex interpolating polynomials. Other important results on interpolation may be found in the papers [**?**] and [**?**] of Szabados, Varga, et al.

II. A consequence of Jensen's formula

For an entire function f and $r > 0$, we define $n_f(r)$ to be the number of zeros of f (counting multiplicities) in the disk $\{z \in \mathbb{C} : |z| \le r\}$ and

$$M_f(r) = \sup\{|f(z)| : |z| = r\}.$$

Proposition II.1 *Let $\Omega > 0$ and $f \in PW(\Omega)$. Suppose $\{z_k\}_{k\in\mathbb{N}}$ are the zeros of f listed with their multiplicities and are arranged so that $|z_1| \le |z_2| \le \cdots$. Suppose*

$$\lim_{r\to+\infty} \frac{n_f(r)}{r} = +\infty.$$

Then $f \equiv 0$.

Proof. Suppose $f \not\equiv 0$. Note that since f is in $PW(\Omega)$, it is entire. Let $N \in \{0,1,2,\dots\}$ be the order of the zero of f at zero. Then there exists an entire function g such that $g(0) \ne 0$ and $f(z) = z^N g(z)$ for all $z \in \mathbb{C}$. Therefore, a consequence [**?**, p. 332] of Jensen's formula is that

$$n_g(r)\log 2 \le \log M_g(2r) \text{ for each } r > 0. \qquad \text{(II.1)}$$

Meanwhile for $\rho > 0$, we have $n_g(\rho) = n_f(\rho) - N$ and $M_g(\rho) = \rho^{-N} M_f(\rho)$. Moreover, the inversion formula

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} \hat{f}(\zeta) e^{i\zeta z}\, d\zeta \quad \forall z \in \mathbb{C},$$

implies that $M_f(\rho) \le (2\pi)^{-1/2}\|\hat{f}\|_1 e^{\Omega\rho}$ for all $\rho > 0$. Therefore in view of (**??**), we obtain

$$(n_f(r) - N)\log 2 \le -N\log 2r + \log\frac{\|\hat{f}\|_1}{\sqrt{2\pi}} + 2r\Omega$$

for $r > 0$. Since $\lim_{r\to\infty} n_f(r) r^{-1} = +\infty$, we have $\|\hat{f}\|_1 = +\infty$, which is absurd. Thus $f \equiv 0$. □

Remark II.2 This proposition implies that if $\{z_k\}_{k=1}^{\infty}$ is a sequence of complex numbers such that

$$\lim_{r\to+\infty} r^{-1}|\{k \in \mathbb{N} : |z_k| \le r\}| = +\infty, \qquad \text{(II.2)}$$

then for any $\Omega > 0$, any f in $PW(\Omega)$ is completely determined by the sampling $\{f(z_k)\}_{k=1}^{\infty}$.

For instance, if $\{z_k\}_{k=1}^{\infty}$ is a bounded sequence, then (**??**) is satisfied. Also, if $(k-1)^\alpha \le |z_k| \le k^\alpha$ $(\forall k \in \mathbb{N})$, for some $\alpha \in]0,1[$, then (**??**) is also satisfied. For these two cases, we will show that the Lagrange interpolation of bandlimited functions converges uniformly on compact subsets of $\mathbb{C}$.

III. Identities involving Lagrange interpolation

We fix a triangular array

$$\{z_{n,k} : k \in \{1,\cdots,n\},\ n \in \mathbb{N}\}$$

of complex numbers such that $z_{n,1},\dots,z_{n,n}$ are distinct for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. For $k \in \{1,2,\cdots,n\}$ we define

$$\ell_{n-1,k}(z) = \left(\prod_{\substack{j=1\\ j\ne k}}^{n}(z_{n,k} - z_{n,j})\right)^{-1} \prod_{\substack{j=1\\ j\ne k}}^{n}(z - z_{n,j}),$$

which are polynomials of degree at most $n-1$ such that $\ell_{n-1,k}(z_{n,k}) = 1$ and $\ell_{n-1,k}(z_{n,j}) = 0$ if $j \ne k$. For a function f defined at the points $z_{n,1},\cdots,z_{n,n}$, we define

$$L_{n-1}f(z) = \sum_{k=1}^{n} f(z_{n,k})\ell_{n-1,k}(z). \qquad \text{(III.1)}$$

Then $L_{n-1}f(z_{n,k}) = f(z_{n,k})$ for $1 \le k \le n$. $L_{n-1}f$ is called the Lagrange polynomial of degree at most $n-1$ which interpolates f at the n points $z_{n,1},\dots,z_{n,n}$.

Our results make use of the following classical formula.

Lemma III.1 *Fix $n \in \mathbb{N}$ and $R > 0$. Suppose $z, z_{n,1}, z_{n,2}, \cdots, z_{n,n}$ are distinct complex numbers contained in the ball $B(0,R) = \{\zeta \in \mathbb{C} : |\zeta| < R\}$ and h is analytic in $B(0,R')$ for some $R' > R$. Then*

$$L_{n-1}h(z) = h(z) + \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{h(\zeta)p_n(z)}{(z-\zeta)p_n(\zeta)}\, d\zeta$$

where $p_n(\cdot) = \prod_{k=1}^{n}(\cdot - z_{n,k})$.

IV. Lagrange interpolation on a bounded set of nodes of entire functions

The following is one of the main results of this paper.

Theorem IV.1 *Let $\Lambda = \{z_{n,k} : k \in \{1,\dots,n\}, n \in \mathbb{N}\}$ be a bounded array of complex numbers such that $z_{n,1},\dots,z_{n,n}$ are distinct for each $n \in \mathbb{N}$. Set*

$\sigma = \sup_{z\in\Lambda}|z|$. *Let f be entire. Then for each $\rho > 0$,*

$$\lim_{n\to\infty}\sup_{|z|\le\rho}|L_{n-1}f(z)-f(z)|^{1/n} = 0.$$

Consequently, $\{L_n f\}_{n=1}^{\infty}$ converges uniformly to f on each compact subset of the complex plane.

Proof. Suppose $f(\omega) = \sum_{j=0}^{\infty} a_j\omega^j$ for all $\omega \in \mathbb{C}$. Let $\rho > \sigma$ and $\varepsilon > 0$. Choose $\delta > 0$ such that

$$\frac{2\delta\rho(\rho+\sigma)}{2\rho-\sigma} < \varepsilon \quad \text{and} \quad 4\delta\rho < 1. \qquad \text{(IV.1)}$$

Since f is entire, there exists $N \in \mathbb{N}$ such that $|a_n| < \delta^n$ whenever $n \ge N$.

Now, fix $n \ge N$ and $|z| \le \rho$. We write $L_{n-1}f(z) - f(z) = R^{(1)}_{n-1}(z) + R^{(2)}_{n-1}(z)$ where

$$\begin{aligned} R^{(1)}_{n-1}(z) &= \sum_{k=1}^{n}\ell_{n-1,k}(z)\sum_{j=0}^{n-1}a_j z^j_{n,k} - \sum_{j=0}^{n-1}a_j z^j \\ &= \sum_{j=0}^{n-1}a_j\left(\sum_{k=1}^{n}\ell_{n-1,k}(z)z^j_{n,k} - z^j\right). \qquad \text{(IV.2)} \end{aligned}$$

Since each inner sum in (**??**) is a polynomial of degree at most $n-1$ that vanishes at the n distinct points $z_{n,1},\dots,z_{n,n}$, it is identically zero. Thus,

$$\begin{aligned} L_{n-1}f(z) - f(z) &= R^{(2)}_{n-1}(z) \\ &= \sum_{j=n}^{\infty}a_j\left(\sum_{k=1}^{n}\ell_{n-1,k}(z)z^j_{n,k} - z^j\right). \end{aligned}$$

We then apply Lemma **??** with $h(\zeta) = \zeta^j$ and $R = 2\rho$ for each $j \ge n$. As a result, we get

$$L_{n-1}f(z) - f(z) = \frac{1}{2\pi i}\sum_{j=n}^{\infty}a_j\int_{|\zeta|=2\rho}\frac{\zeta^j p_n(z)}{(z-\zeta)p_n(\zeta)}\,d\zeta, \qquad \text{(IV.3)}$$

with $p_n(\cdot) = \prod_{k=1}^{n}(\cdot - z_{n,k})$. Meanwhile

$$\left|\frac{p_n(z)}{p_n(\zeta)}\right| \le \left(\frac{\rho+\sigma}{2\rho-\sigma}\right)^n$$

for $|\zeta| = 2\rho$. Hence

$$\begin{aligned} |L_{n-1}f(z) - f(z)| &\le 2\left(\frac{\rho+\sigma}{2\rho-\sigma}\right)^n\sum_{j=n}^{\infty}(2\rho\delta)^j \\ &\le 2\left(\frac{\rho+\sigma}{2\rho-\sigma}\right)^n\frac{(2\rho\delta)^n}{1-2\rho\delta}. \end{aligned}$$

Finally, combining this with (**??**), we obtain

$$|L_{n-1}f(z) - f(z)| \le 4\varepsilon^n.$$

□

V. Lagrange interpolation of bandlimited functions on slowly increasing nodes

Another of our main results is the following theorem.

Theorem V.1 *Let $\Omega > 0$, $\alpha \in]0,1[$, and suppose $\{z_k\}_{k=1}^{\infty}$ is a sequence of distinct complex numbers such that*

$$(k-1)^\alpha \le |z_k| \le k^\alpha \quad \forall k \in \mathbb{N}. \qquad \text{(V.1)}$$

Let $g \in L^1([-\Omega,\Omega])$ and define an entire function f by

$$f(z) = \int_{-\Omega}^{\Omega} g(t)e^{itz}dt \quad \forall z \in \mathbb{C}.$$

Then for each $\rho > 1$, there exists $N_0 \in \mathbb{N}$ such that for all $n \ge N_0$,

$$\sup_{|z|\le\rho}|L_{n-1}f(z) - f(z)| \le C\|g\|_1 Q(\rho)A^n(n+1)^{-p_n}, \qquad \text{(V.2)}$$

where

$$Q(\rho) = \rho^{-1/2}\exp(\alpha\rho^{1/\alpha}), \quad p_n = n(1-\alpha)+1-\frac{\alpha}{2}, \qquad \text{(V.3)}$$

and A and C are constants depending only on α and Ω: $A = 2\alpha^{-1}\Omega\exp(2+\Omega-\alpha)$ and $C = \pi^{-1}2^{\alpha/2}e^{2+\Omega}$.

Proof. Let $f(\zeta) = \sum_{j=0}^{\infty}a_j\zeta^j$ for each $\zeta \in \mathbb{C}$. Let $\rho > 1$ and let N be the integer satisfying $(N-1)^\alpha < \rho \le N^\alpha$. (Necessarily, $N \ge 2$.) Now, fix $n \ge 2^{1/\alpha}N$ and $|z| \le \rho$. Applying Lemma **??** with $R = (n+1)^\alpha$, we obtain as in the proof of (**??**),

$$L_{n-1}f(z) - f(z) = \frac{1}{2\pi i}\sum_{j=n}^{\infty}a_j\int_{|\zeta|=(n+1)^\alpha}\frac{\zeta^j p_n(z)}{(z-\zeta)p_n(\zeta)}\,d\zeta \qquad \text{(V.4)}$$

where $p_n(\cdot) = \prod_{k=1}^{n}(\cdot - z_k)$.

Meanwhile, for $|\zeta| = (n+1)^\alpha$, we have

$$|p_n(\zeta)| \ge (n+1)^{n\alpha}\prod_{k=1}^{n}\left(1-\left(\frac{k}{n+1}\right)^\alpha\right).$$

Applying the inequality $1 - t^\alpha \ge \alpha(1-t)$, $0 < t < 1$, with $t = \frac{k}{n+1}$ $(1 \le k \le n)$, we obtain

$$|p_n(\zeta)| \ge \frac{\alpha^n n!}{(n+1)^{n(1-\alpha)}} \quad \text{for } |\zeta| = (n+1)^\alpha. \qquad \text{(V.5)}$$

Next, we turn to $|p_n(z)|$. Since $|z| < \rho$ and $0 < \alpha < 1$, our choice of N ($(N-1)^\alpha < \rho$) and (**??**) imply, on one hand that

$$\prod_{k=1}^{N-1}|z - z_k| \le (2\rho)^{N-1}. \qquad \text{(V.6)}$$

On the other hand, since $\rho \le N^\alpha$, we obtain

$$\prod_{k=N}^{n}|z - z_k| \le \prod_{k=N}^{n}(2k^\alpha) = \frac{2^{n-N+1}(n!)^\alpha}{((N-1)!)^\alpha}. \qquad \text{(V.7)}$$

Combining (**??**) and (**??**) gives

$$|p_n(z)| \ \le \ \frac{2^n \rho^{N-1}(n!)^\alpha}{((N-1)!)^\alpha}. \tag{V.8}$$

Thus, we conclude from (**??**) and (**??**) that

$$\left|\frac{p_n(z)}{p_n(\zeta)}\right| \ \le \ \theta_n Q_1(\rho) \tag{V.9}$$

whenever $|\zeta| = (n+1)^\alpha$, where

$$\theta_n = \left(\frac{2}{\alpha}\right)^n \left(\frac{(n+1)^n}{n!}\right)^{1-\alpha} \text{and } Q_1(\rho) = \frac{\rho^{N-1}}{((N-1)!)^\alpha}. \tag{V.10}$$

Meanwhile by Stirling's formula [**?**, p.204],

$$n! \ge \sqrt{2\pi} e^{-n-1}(n+1)^{n+\frac{1}{2}} \quad \text{for} \quad n \in \mathbb{N}. \tag{V.11}$$

This provides the following estimate for θ_n:

$$\theta_n \ \le \ \left(\frac{2}{\alpha}\right)^n \left(\frac{e^{n+1}}{\sqrt{2\pi(n+1)}}\right)^{1-\alpha}. \tag{V.12}$$

Since $\rho > 1$, then in view of (**??**) and the conditions $(N-1)^\alpha < \rho \le N^\alpha$, we obtain the following estimates for $Q_1(\rho)$:

$$\begin{aligned} Q_1(\rho) &\le \left(\frac{1+\rho^{1/\alpha}}{2\pi}\right)^{\alpha/2} \frac{\exp(\alpha + \alpha\rho^{1/\alpha})}{\rho} \\ &\le \left(\frac{e}{\sqrt{\pi}}\right)^\alpha \frac{\exp(\alpha\rho^{1/\alpha})}{\rho^{1/2}}. \end{aligned} \tag{V.13}$$

Meanwhile, if $|\zeta| = (n+1)^\alpha$, then

$$|\zeta - z| \ge (n+1)^\alpha - \rho \ge \frac{1}{2}(n+1)^\alpha.$$

Substituting this, (**??**), and the estimates $|a_j| \le (j!)^{-1}\|g\|_1 \Omega^j$ $(\forall j \in \mathbb{N})$ into (**??**), we get

$$|L_{n-1}f(z) - f(z)| \le 2\|g\|_1 \theta_n \ Q_1(\rho) \sum_{j=n}^{\infty} \frac{(\Omega(n+1)^\alpha)^j}{j!}.$$

Applying the estimate

$$\sum_{j=n}^{\infty} \frac{u^j}{j!} \ \le \ \frac{u^n e^u}{n!} \qquad \text{with } u = \Omega(n+1)^\alpha,$$

(**??**), (**??**) and again (**??**), we finally obtain (**??**). □

The next proposition shows that the estimate (**??**) is optimal in relation to the exponent α in (**??**) and in (**??**).

Proposition V.2 *Let $\{z_k\}_{k=1}^\infty$ be a sequence of distinct nonzero complex numbers such that $\{|z_k|\}_{k=1}^\infty$ is increasing, and let f be an entire function. For each $n \in \mathbb{N}$, define $L_{n-1}f$ as in* (**??**) *(with $z_{n,k} = z_k$ for each $n \in \mathbb{N}$ and for each $k \in \{1, 2, \cdots, n\}$). Let $\alpha \in \,]0,1[$, $\{\varepsilon_n\}_{n=1}^\infty \subset \,]0,+\infty[$, and C, C_1 be positive constants such that for each $\rho > 0$,*

$$\sup_{|z|\le\rho} |L_{n-1}f(z) - f(z)| \ \le \ C\exp(\alpha\rho^{1/\alpha})\varepsilon_n \tag{V.14}$$

and $|L_{n-1}f(0) - f(0)| \ge C_1\varepsilon_n$ for each $n \in \mathbb{N}$. Then there exists a constant $C_0 > 0$ such that

$$|z_n| \ \ge \ C_0 n^\alpha, \quad \textit{for all } n \in \mathbb{N}.$$

Proof. Fix $n \in \mathbb{N}$ and set $r = |z_n|$. Define $g = (L_{n-1}f(0) - f(0))^{-1}(L_{n-1}f - f)$. Applying (**??**) with $\rho = 2r$ gives

$$M_g(2r) \ \le \ \frac{C}{C_1} \exp\left(\alpha(2r)^{1/\alpha}\right).$$

Finally, since $n_g(r) = n$, we obtain from (**??**) that

$$n \log 2 \ \le \ \log\frac{C}{C_1} \ + \ \alpha(2|z_n|)^{1/\alpha}.$$

This furnishes the desired estimate. □

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