Lagrange interpolation of bandlimited functions on slowly increasing sequences

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Abstract—Let \( \Lambda = \{ z_{n,k} : 1 \leq k \leq n, \; n \in \mathbb{N} \} \) be a triangular array of distinct complex numbers and let \( f \) be an entire function. Suppose \( L_n f \) is the unique polynomial of degree at most \( n - 1 \) which interpolates \( f \) at \( z_{n,k} \) for \( k \in \{ 1, \cdots, n \} \), i.e., \( L_n f(z_{n,k}) = f(z_{n,k}) \). In this note, we show that \( L_n f \) converges to \( f \) uniformly on compact subsets of the complex plane provided \( \Lambda \) is bounded. We next consider the case when \( z_{n,k} = z_k \) where \( \{ z_k \}_{k \in \mathbb{N}} \) is a slowly increasing unbounded sequence in the sense that for some \( \alpha \in [0,1] \), \( (k-1)\alpha \leq |z_k| \leq k^\alpha \) for each \( k \in \mathbb{N} \). If \( f \) is bandlimited, we prove as well that \( L_n f \) converges uniformly (and rapidly) to \( f \) on compact subsets of the complex plane. The rate of convergence that we obtain is optimal to some extent.

Keywords: Lagrange interpolation; signal recovery; sampling; bandlimited signals; entire functions; Jensen’s formula

I. INTRODUCTION

Fix \( \Omega > 0 \) and let \( PW(\Omega) \) denote the Paley-Wiener class, that is, the set of functions \( f \in L^2(\mathbb{R}) \) whose Fourier transform
\[
\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)e^{-i\omega t} dt
\]
is supported on \([ -\Omega, \Omega ] \). Suppose \( 0 < \delta \Omega < \pi \) and \( \phi \) is a function from the Schwartz class with \( \hat{\phi} \) supported on \([ -\pi \delta^{-1}, \pi \delta^{-1} ] \) and such that \( \hat{\phi} = (2\pi)^{-1/2} \) on \([ -\Omega, \Omega ] \). Then any \( f \) in \( PW(\Omega) \) can be recovered from its values \( \{ f(k\delta) : k \in \mathbb{Z} \} \) by the formula
\[
f(x) = \delta \sum_{k \in \mathbb{Z}} f(k\delta)\phi(x - k\delta)
\]
[?].

More generally, let \( \{ x_k \}_{k=\infty}^{\infty} \) be a bi-infinite increasing sequence in \( \mathbb{R} \), such that \( \lim_{k \to \pm \infty} x_k = \pm \infty \) and \( \Delta = \sup_{k \in \mathbb{Z}}(x_k - x_{k-1}) < \infty \). If \( f \in PW(\Omega) \) with \( \Delta \Omega < \pi \), it is well known that we may recover \( f \) from the samples \( \{ f(x_k) \}_{k=-\infty}^{\infty} \). In fact, in [?] and [?], the authors provide algorithms for signal reconstruction.

We pose the following question: if \( \{ t_k \}_{k=1}^{\infty} \) is a sequence in \([ 0, +\infty ) \) slowly increasing to infinity, is it possible to recover \( f \in PW(\Omega) \) from its samples \( \{ f(t_k) \}_{k=1}^{\infty} \)? In this note, we prove that if for some \( \alpha \in [0,1] \), \( (k-1)\alpha \leq |z_k| \leq k^\alpha \) for all \( k \in \mathbb{N} \), then the Lagrange polynomials \( L_n f \) interpolating \( f \) at the points \( z_1, \cdots, z_n \) converge uniformly to \( f \) on compact subsets of the complex plane. The rate of convergence that we obtain is optimal to some extent.

This result answers a special case of a more general problem. Let \( \{ z_k \}_{k=1}^{\infty} \) be a sequence of complex numbers satisfying
\[
\lim_{r \to +\infty} r^{-1} | \{ k \in \mathbb{N} : |z_k| \leq r \} | = +\infty. \tag{1.1}
\]

Then, any \( f \) in \( PW(\Omega) \) is completely determined by its samples \( \{ f(z_k) \}_{k=1}^{\infty} \) (Proposition ?? below). Thus, an interesting task is to reconstruct \( f \) from \( \{ f(z_k) \}_{k=1}^{\infty} \). When \( \{ z_k \}_{k=1}^{\infty} \) is bounded, (??) is satisfied and we show in Theorem ?? that Lagrange interpolation accomplishes the reconstruction. Moreover, if \( (k-1)\alpha \leq |z_k| \leq k^\alpha \) with \( 0 < \alpha < 1 \), the condition in (??) is also satisfied and we also show that Lagrange interpolation does the job in Theorem ??.

One of the most beautiful results on Lagrange interpolation goes back to the 1930’s by J.L. Walsh. Suppose \( f \) is analytic on the disk \( \{ z \in \mathbb{C} : |z| < \rho \} \) for some \( \rho > 1 \). For a positive integer \( n \), let \( L_n f \) be the unique polynomial of degree at most \( n - 1 \) which interpolates \( f \) at the zeros of \( z^n - 1 \). Walsh proved that
\[
\lim_{n \to \infty} \left( L_n f(z) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} z^k \right) = 0
\]
uniformly for \( |z| \leq r \) whenever \( 0 < r < \rho^2 \). This work is also inspired by results of Totik in [?] on the recovery of \( H^p \) functions on the open unit disk \( \mathbb{D} = \)
Proposition II.1 Let \( \Omega > 0 \) and \( f \in PW(\Omega) \). Suppose \( \{z_k\}_{k \in \mathbb{N}} \) are the zeros of \( f \) listed with their multiplicities and are arranged so that \( |z_1| \leq |z_2| \leq \cdots \). Suppose

\[
\lim_{r \to +\infty} \frac{n_f(r)}{r} = +\infty.
\]

Then \( f \equiv 0 \).

Proof. Suppose \( f \neq 0 \). Note that since \( f \) is \( PW(\Omega) \), it is entire. Let \( N \in \{0, 1, 2, \ldots\} \) be the order of the zero of \( f \) at zero. Then there exists an entire function \( g \) such that \( g(0) \neq 0 \) and \( f(z) = z^NG(z) \) for all \( z \in \mathbb{C} \). Therefore, a consequence [?, p. 332] of Jensen’s formula is that

\[
n_g(r) \log 2 \leq \log M_g(2r) \quad \text{for each } r > 0. \tag{II.1}
\]

Meanwhile for \( \rho > 0 \), we have \( n_g(\rho) = n_f(\rho) - N \) and \( M_g(\rho) = \rho^{-N}M_f(\rho) \). Moreover, the inversion formula

\[
f(z) = \frac{1}{2\pi i} \int_{-\Omega}^{\Omega} \hat{f}(\zeta) e^{\zeta z} d\zeta \quad \forall z \in \mathbb{C},
\]

implies that \( M_f(\rho) \leq (2\pi)^{-1/2}\|\hat{f}\|_1 e^\Omega\rho \) for all \( \rho > 0 \). Therefore in view of (II.1), we obtain

\[
(n_f(r) - N) \log 2 \leq -N \log 2r + \log \frac{\|\hat{f}\|_1}{\sqrt{2\pi}} + 2r\Omega
\]

for \( r > 0 \). Since \( \lim_{r \to \infty} n_f(r)r^{-1} = +\infty \), we have \( \|\hat{f}\|_1 = +\infty \), which is absurd. Thus \( f \equiv 0 \). \( \square \)

Remark II.2 This proposition implies that if \( \{z_k\}_{k=1}^{\infty} \) is a sequence of complex numbers such that

\[
\lim_{r \to +\infty} r^{-1}|\{k \in \mathbb{N} : |z_k| \leq r\}| = +\infty, \tag{II.2}
\]

then for any \( \Omega > 0 \), any \( f \) in \( PW(\Omega) \) is completely determined by the sampling \( \{f(z_k)\}_{k=1}^{\infty} \).

For instance, if \( \{z_k\}_{k=1}^{\infty} \) is a bounded sequence, then (II.2) is satisfied. Also, if \( (k - 1)^\alpha \leq |z_k| \leq k^\alpha \) for some \( \alpha \in [0, 1] \), then (II.2) is also satisfied. For these two cases, we will show that the Lagrange interpolation of bandlimited functions converges uniformly on compact subsets of \( \mathbb{C} \).

III. IDENTITIES INVOLVING LAGRANGE INTERPOLATION

We fix a triangular array

\[
\{z_{n,k} : k \in \{1, \ldots, n\}, n \in \mathbb{N}\}
\]

of complex numbers such that \( z_{n,1}, \ldots, z_{n,n} \) are distinct for every \( n \in \mathbb{N} \). Fix \( n \in \mathbb{N} \). For \( k \in \{1, 2, \ldots, n\} \) we define

\[
\ell_{n-1,k}(z) = \left( \prod_{j=1}^{n} (z_{n,k} - z_{n,j}) \right)^{-1} \prod_{j=1}^{n} (z - z_{n,j}),
\]

which are polynomials of degree at most \( n - 1 \) such that \( \ell_{n-1,k}(z_{n,k}) = 1 \) and \( \ell_{n-1,k}(z_{n,j}) = 0 \) if \( j \neq k \). For a function \( f \) defined at the points \( z_{n,1}, \ldots, z_{n,n} \), we define

\[
L_{n-1}f(z) = \sum_{k=1}^{n} f(z_{n,k})\ell_{n-1,k}(z). \tag{III.1}
\]

Then \( L_{n-1}f(z_{n,k}) = f(z_{n,k}) \) for \( 1 \leq k \leq n \). \( L_{n-1}f \) is called the Lagrange polynomial of degree at most \( n - 1 \) which interpolates \( f \) at the \( n \) points \( z_{n,1}, \ldots, z_{n,n} \).

Our results make use of the following classical formula.

**Lemma III.1** Fix \( n \in \mathbb{N} \) and \( R > 0 \). Suppose \( z, z_{n,1}, z_{n,2}, \ldots, z_{n,n} \) are distinct complex numbers contained in the ball \( B(0,R) = \{\zeta \in \mathbb{C} : |\zeta| < R\} \) and \( h \) is analytic in \( B(0,R') \) for some \( R' > R \). Then

\[
L_{n-1}h(z) = h(z) + \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{h(\zeta)p_n(z)}{(z - \zeta)p_n(\zeta)} d\zeta
\]

where \( p_n(\cdot) = \prod_{k=1}^{n}(\cdot - z_{n,k}) \).

IV. LAGRANGE INTERPOLATION ON A BOUNDED SET OF NODES OF ENTIRE FUNCTIONS

The following is one of the main results of this paper.

**Theorem IV.1** Let \( \Lambda = \{z_{n,k} : k \in \{1, \ldots, n\}, n \in \mathbb{N}\} \) be a bounded array of complex numbers such that \( z_{n,1}, \ldots, z_{n,n} \) are distinct for each \( n \in \mathbb{N} \). Set
σ = \sup_{z \in A} |z|. Let f be entire. Then for each ρ > 0,
\lim_{n \to \infty} \sup_{|z| \leq ρ} |L_{n-1} f(z) - f(z)|^{1/n} = 0.

Consequently, \{L_n f\}_n converges uniformly on each compact subset of the complex plane.

Proof. Suppose f(ω) = \sum_{j=0}^{∞} a_j ω^j for all ω ∈ C. Let ρ > σ and ε > 0. Choose δ > 0 such that
\frac{2δρ(ρ + σ)}{2ρ - σ} < ε and 4δρ < 1. (IV.1)

Since f is entire, there exists N ∈ N such that |a_n| < δ^n whenever n ≥ N.

Now, fix n ≥ N and |z| ≤ ρ. We write
\begin{align*}
L_{n-1} f(z) - f(z) = R^{(1)}_{n-1}(z) + R^{(2)}_{n-1}(z)
\end{align*}
where
\begin{align*}
R^{(1)}_{n-1}(z) &= \sum_{k=1}^{n} \ell_{n-1,k}(z) \sum_{j=0}^{n-1} a_j z^j \sum_{j=0}^{n-1} a_j z^j \\
&= \sum_{j=0}^{n-1} a_j \left( \sum_{k=1}^{n} \ell_{n-1,k}(z) z^j \right) - z^{j}.
\end{align*}

Since each inner sum in (??) is a polynomial of degree at most n - 1 that vanishes at the n distinct points z_{1,j}, ..., z_{n,n}, it is identically zero. Thus,
\begin{align*}
L_{n-1} f(z) - f(z) = R^{(2)}_{n-1}(z) = \sum_{j=0}^{n} a_j \left( \sum_{k=1}^{n} \ell_{n-1,k}(z) z^j \right) - z^{j}.
\end{align*}

We then apply Lemma ?? with h(z) = ζ^j and R = 2ρ for each j ≥ n. As a result, we get
\begin{align*}
L_{n-1} f(z) - f(z) = \frac{1}{2πi} \sum_{j=0}^{∞} a_j \int_{|ζ|=2ρ} \frac{ζ^j p_n(ζ)}{(ζ - ζ)p_n(ζ)} dζ,
\end{align*}
with p_n(ζ) = \prod_{k=1}^{n} (ζ - z_{k,n}). Meanwhile
\|p_n(ζ)\| \leq \left( \frac{ρ + σ}{2ρ - σ} \right)^n
for |ζ| = 2ρ. Hence
\begin{align*}
|L_{n-1} f(z) - f(z)| \leq 2 \left( \frac{ρ + σ}{2ρ - σ} \right)^n (2ρδ)^j
\leq 2 \left( \frac{ρ + σ}{2ρ - σ} \right)^n (2ρδ)^n \left( \frac{2ρδ}{1 - 2ρδ} \right).
\end{align*}

Finally, combining this with (??), we obtain
\begin{align*}
|L_{n-1} f(z) - f(z)| \leq 4ε^n.
\end{align*}

V. LAGRANGE INTERPOLATION OF BANDLIMITED FUNCTIONS ON SLOWLY INCREASING NODES

Another of our main results is the following theorem.

Theorem V.1 Let Ω > 0, α ∈ [0, 1], and suppose \{z_k\}_k is a sequence of distinct complex numbers such that
\begin{align*}
(k - 1)^α \leq |z_k| \leq k^α \quad \forall k ∈ N.
\end{align*}

Let g ∈ L^1(−Ω, Ω) and define an entire function f by
\begin{align*}
f(z) = \int_{-Ω}^{Ω} g(t) e^{iζt} dt \quad ∀ζ ∈ C.
\end{align*}

Then for each ρ > 1, there exists N_0 ∈ N such that for all n ≥ N_0,
\begin{align*}
\sup_{|ζ| ≤ ρ} |L_n f(z) - f(ζ)| ≤ C ||g||_1 Q(ρ) A_n (n + 1)^{-p_n},
\end{align*}
where
\begin{align*}
Q(ρ) = ρ^{-1/2} \exp(αρ^{1/α}) \quad p_n = n(1 - α) + 1 - \frac{α}{2},
\end{align*}
and A and C are constants depending only on α and Ω:
\begin{align*}
A = 2α^{-1} Ω \exp(2 + Ω - α) \quad C = π^{-1} 2^{α/2} e^{2πΩ}.
\end{align*}

Proof. Let f(ζ) = \sum_{j=0}^{∞} a_j ζ^j for each ζ ∈ C. Let ρ > 1 and let N be the integer satisfying (N - 1)^α < ρ ≤ N^α. (Necessarily, N ≥ 2.) Now, fix n ≥ 2^{1/α} N and |ζ| ≤ ρ. Applying Lemma ?? with R = (n + 1)^α, we obtain as in the proof of (??),
\begin{align*}
L_{n-1} f(z) - f(z) = \frac{1}{2πi} \sum_{j=n}^{∞} a_j \int_{|ζ|=|ζ|=2ρ} \frac{ζ^j p_n(ζ)}{(ζ - ζ)p_n(ζ)} dζ,
\end{align*}
where p_n(ζ) = \prod_{k=1}^{N} (ζ - z_{k}).

Meanwhile, for |ζ| = (n + 1)^α, we have
\begin{align*}
|p_n(ζ)| \geq (n + 1)^{n} \prod_{k=1}^{n} \left( 1 - \frac{k}{n + 1} \right)^α
\end{align*}
Applying the inequality 1 - t^α ≥ α(1 - t), 0 < t < 1, with t = k/(n + 1) (1 ≤ k ≤ n), we obtain
\begin{align*}
|p_n(ζ)| \geq \frac{α^n n!}{(n + 1)^{n(1+α)}} \quad \text{for } |ζ| = (n + 1)^α.
\end{align*}

Next, we turn to |p_n(ζ)|. Since |ζ| < ρ and 0 < α < 1, our choice of N ( (N - 1)^α < ρ ) and (??) imply, on one hand that
\begin{align*}
N^{-1} \prod_{k=1}^{N-1} |ζ - z_k| \leq (2ρ)^{N-1}.
\end{align*}
On the other hand, since ρ ≤ N^α, we obtain
\begin{align*}
\prod_{k=N}^{n} |ζ - z_k| \leq \prod_{k=N}^{n} (2k^α) = \frac{2^{n-N+1}(n+1)^α}{((N-1)!)^α}.
\end{align*}
Combining (19) and (20) gives
\[ |p_n(z)| \leq \frac{2n^n \rho N^{-1}(n!)^\alpha}{((N-1)!)^\alpha}. \] (V.8)
Thus, we conclude from (21) and (22) that
\[ \left| \frac{p_n(z)}{p_n(\zeta)} \right| \leq \theta_n Q_1(\rho) \] (V.9)
whenever \(|z| = (n+1)^\alpha\), where
\[ \theta_n = \left( \frac{2}{\alpha} \right)^n \left( \frac{(n+1)^n}{n!} \right)^{1-\alpha} \]
and
\[ Q_1(\rho) = \frac{\rho N^{-1}}{((N-1)!)^\alpha}. \]

Meanwhile by Stirling’s formula [?, p.204],
\[ n! \geq \sqrt{2\pi e^{-n-1}(n+1)^n} \frac{1}{2} \] for \( n \in \mathbb{N}. \) (V.11)
This provides the following estimate for \( \theta_n \):
\[ \theta_n \leq \left( \frac{2}{\alpha} \right)^n \left( \frac{e^{n+1}}{\sqrt{2\pi(n+1)}} \right)^{1-\alpha}. \] (V.12)
Since \( \rho > 1 \), then in view of (23) and the conditions \((N-1)^\alpha < \rho \leq N^\alpha\), we obtain the following estimates for \( Q_1(\rho) \):
\[ Q_1(\rho) \leq \left( \frac{1 + \rho^{1/\alpha}}{2\pi} \right)^{\alpha/2} \exp(\alpha + \alpha\rho^{1/\alpha}) \]
\[ \leq \left( \frac{e}{\sqrt{\pi}} \right)^\alpha \frac{\exp(\alpha\rho^{1/\alpha})}{\rho^{1/2}}. \] (V.13)
Meanwhile, if \(|z| = (n+1)^\alpha\), then
\[ |z - \zeta| \geq (n+1)^\alpha - \rho \geq \frac{1}{2}(n+1)^\alpha. \]
Substituting this, (24), and the estimates \(|a_j| \leq (j!)^{-1}\|g\|_1\Omega^j (\forall j \in \mathbb{N})\) into (25), we get
\[ |L_{n-1}f(z) - f(z)| \leq 2\|g\|_1 \theta_n Q_1(\rho) \sum_{j=n}^{\infty} \frac{(\Omega(n+1)^\alpha)^j}{j!}. \]
Applying the estimate
\[ \sum_{j=n}^{\infty} \frac{u^j}{j!} \leq \frac{u^n e^u}{n!} \] with \( u = \Omega(n+1)^\alpha, \)
(26), (27) and again (28), we finally obtain (29). \( \square \)

The next proposition shows that the estimate (29) is optimal in relation to the exponent \( \alpha \) in (24) and in (25).

**Proposition V.2** Let \( \{z_k\}_{k=1}^{\infty} \) be a sequence of distinct nonzero complex numbers such that \( \{|z_k|\}_{k=1}^{\infty} \) is increasing, and let \( f \) be an entire function. For each \( n \in \mathbb{N} \), define \( L_{n-1}f \) as in (22) (with \( z_{n,k} = z_k \) for each \( n \in \mathbb{N} \) and for each \( k \in \{1, 2, \ldots, n\} \)). Let \( \alpha \in [0, 1] \), \( \varepsilon_n \}_{n=1}^{\infty} \subset ]0, +\infty[ \), and \( C, C_1 \) be positive constants such that for each \( \rho > 0 \),
\[ \sup_{|z| \leq \rho} |L_{n-1}f(z) - f(z)| \leq C \exp(\alpha \rho^{1/\alpha}) \varepsilon_n \] (V.14)
and \( |L_{n-1}f(0) - f(0)| \geq C \varepsilon_n \) for each \( n \in \mathbb{N} \). Then there exists a constant \( C_0 > 0 \) such that
\[ |z_n| \geq C_0 n^\alpha, \] for all \( n \in \mathbb{N}. \)

**Proof.** Fix \( n \in \mathbb{N} \) and set \( r = |z_n| \). Define \( g = (L_{n-1}f(0) - f(0))^{-1}(L_{n-1}f - f) \). Applying (29) with \( \rho = 2r \) gives
\[ M_g(2r) \leq \frac{C}{C_1} \exp \left( \alpha(2r)^{1/\alpha} \right). \]
Finally, since \( n_g(r) = n \), we obtain from (30) that
\[ n \log 2 \leq \log \frac{C}{C_1} + \alpha(2|z_n|)^{1/\alpha}. \]
This furnishes the desired estimate. \( \square \)

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