On the Reconstruction of a Class of Signals Bandlimited to a Disc

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***Abstract*—Signal reconstruction is one of the most important problems in signal processing and sampling theorems are one of the main tools used for such reconstructions. There is a vast literature on sampling in one and higher dimensions of bandlimited signals. Because the sampling formulas and points depend on the geometry of the domain on which the signals are confined, explicit representations of the reconstruction formulas exist mainly for domains that are geometrically simple, such as intervals or parallelepiped symmetric about the origin.**

In this talk we derive sampling theorem for the reconstruction of signals that are bandlimited to a disc centered at the origin. This will be done for a more general class of signals than those that are bandlimited in the Fourier transform domain. The sampling points are related to the zeros of the Bessel function.

***Index Terms*—Two-dimensional sampling theorem, Signal reconstruction.**

I. Introduction

Sampling theorems are important tools in signal processing and communication because they allow the conversion of analog signals into digital signals which can be processed digitally and then converted back. The fundamental paper in this field is Shannon's seminal paper [8] which showed how a signal bandlimited to, say an interval $[-\sigma, \sigma]$, can be sampled and converted to a digital signal and then converted back to an analog signal using shifts of the sinc function $\sin \pi x/\pi x$. Since the publication of Shannon's paper many generalizations have been developed, such as sampling in higher dimensions, sampling with the derivatives, and sampling of functions given by integral transforms other than the Fourier transform.

Sampling theorems may be obtained by using different techniques, among them are orthogonal sampling systems, Zak transform, and reproducing-kernel Hilbert space theory. In [1], [2], Bhandari and Zayed obtained sampling theorems for the special affine Fourier transform from which sampling theorem for the *linear canonical transform* (LCT) may be obtained as a special case. Other results concerning sampling of LCT can be found in [6], [9], [10], [11], [12], [13], [7]; see also [14] for related results. All the above mentioned results deal with signals that are bandlimited to an interval symmetric around the origin, or a rectangle centered at the origin in two dimensions, or a parallelepiped in n dimension. Sampling theorems for signals that are bandlimited to a general domain are challenging to obtain in closed form because the sampling points and functions depend on the geometry of the domain for which the sinc function is not appropriate.

Since the *fractional Fourier transform* (FrFT) is a special case of LCT, sampling theorems for the FrFT are special cases of those for the LCT.

The purpose of this article is to obtain a reconstruction formula for signals that are bandlimited to a disc of radius R in the LCT domain. The generalization of the sampling theorem to a disc requires different techniques than those used for intervals and rectangles and the *sinc* function plays no role in the derivation. As a special case, we obtain sampling formula for signals that are bandlimited to a disc in the fractional Fourier transform domain.

A. The Two-Dimensional LCT

Let $\mathbf{t} = (t_1, \cdots, , t_n), \mathbf{x} = (x_1, \cdots, , x_n)$, $\mathbf{x} \cdot \mathbf{t} = x_1 t_1 + \cdots + x_n t_n$, and $|x|^2 = x_1^2 + \cdots + x_n^2$. The n-dimensional linear canonical transform of a function $f \in L^1(\mathbb{R}^n)$ is defined as

$$F(\mathbf{t}) = \frac{1}{(2\pi b)^{n/2}} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{\frac{i}{2b}\left(a|x|^2 + d|t|^2 - 2\mathbf{x}\cdot\mathbf{t}\right)} d\mathbf{x},$$

where $d\mathbf{x} = dx_1 \cdots dx_n$, and a, b, c, d are real numbers such that $ad - bc = 1$, $b \neq 0$. The fractional Fourier transform is obtained by setting $a = \cos\theta = d, b = \sin\theta = -c$.

In two dimensions the transform can be simplified further:

$$F(\mathbf{t}) = \frac{1}{(2\pi b)} \int_{\mathbb{R}^2} f(\mathbf{x}) e^{\frac{i}{2b}\left(a(|x|^2 + d|t|^2 - 2(x_1 t_1 + x_2 t_2)\right)} dx_1 dx_2.$$

Using polar coordinates

$$x_1 = r\cos\theta,\ x_2 = r\sin\theta,\ t_1 = \rho\cos\phi,\ t_2 = \rho\sin\phi,$$

we obtain

$$F(\rho, \phi) = \frac{1}{(2\pi b)} \int_0^\infty \int_0^{2\pi} f(r, \theta) e^{\frac{i}{2b}\left(ar^2 + d\rho^2 - 2r\rho\cos(\theta - \phi)\right)} r dr d\theta.$$

Hence,

$$\tilde{F}(\rho, \phi) = \frac{1}{2\pi b} \int_0^\infty \int_0^{2\pi} \tilde{f}(r, \theta) e^{-\frac{i}{b} r\rho\cos(\theta - \phi)} r dr d\theta,$$

where

$$\tilde{F}(\rho, \phi) = e^{-id\rho^2/2b} F(\rho, \phi), \text{ and } \tilde{f}(r, \theta) = e^{iar^2/2b} f(r, \theta).$$

In view of the relation [3, p. 973],

$$e^{-it\sin\psi} = \sum_{n=-\infty}^{\infty} J_n(t) e^{-in\psi} \tag{1}$$

we obtain

$$\tilde{F}(\rho, \phi) = \frac{1}{2\pi b} \int_0^\infty \int_0^{2\pi} \tilde{f}(r, \theta) \sum_{n=-\infty}^{\infty} (-i)^n J_n(r\rho/b) e^{-in(\theta - \phi)} r dr d\theta, \tag{2}$$

where $J_\nu(z)$ is the Bessel function of the first kind and order $\nu \geq -1/2$. Let us denote the positive zeros of $J_\nu(z)$ by

$$0 < z_{\nu,1} < z_{\nu,2} < \cdots < z_{\nu,n} < \cdots .$$

From the relation [5, p. 128]

$$\int_0^a rJ_\nu(\alpha r)J_\nu(\beta r)dr = \frac{a\beta J_\nu(\alpha a)J'_\nu(\beta a) - a\alpha J_\nu(\beta a)J'_\nu(\alpha a)}{\alpha^2 - \beta^2}, \tag{3}$$

we obtain by setting $\alpha = \alpha_{\nu,n} = z_{\nu,n}/a$ and $\beta = \ell/a$

$$\int_0^a J_\nu(\alpha_{\nu,n} r)\, J_\nu(\ell r/a)\, r dr = \frac{a^2 z_{\nu,n} J_{\nu+1}(z_{\nu,n})\, J_\nu(\ell)}{z_{\nu,n}^2 - \ell^2}. \tag{4}$$

We also have

$$\int_0^a J_\nu(\alpha_{\nu,n} x)\, J_\nu(\alpha_{\nu,m} x)\, x dx = \begin{cases} 0, & m \neq n \\ \frac{a^2}{2} J_{\nu+1}^2(z_{\nu,n}), & m = n \end{cases} \tag{5}$$

II. The Sampling Theorem

To derive the main result of the paper, we need the following lemmas for which an abridged proof will be given.

Lemma 1. *Consider $J_\nu(\rho x)$ where $0 \leq x \leq a$ and $\rho \geq 0$, and let $\alpha_{\nu n}$, $z_{\nu n}$ be defined as before. Then*

$$J_\nu(\rho x) = \sum_{n=1}^{\infty} \frac{2z_{\nu,n} J_\nu(a\rho) J_\nu(\alpha_{\nu,n} x)}{\left(z_{\nu,n}^2 - a^2\rho^2\right) J_{\nu+1}(z_{\nu,n})}.$$

Proof. By expanding $J_\nu(\rho x)$ in terms of the orthogonal system given by Eq. (5), we have

$$J_\nu(\rho x) = \sum_{n=1}^{\infty} b_n(\rho) J_\nu(\alpha_{\nu,n} x),$$

and by using Eq.(4), and then solving for $b_n(\rho)$, we obtain the result. □

With the aid of Lemma 1 and some easy calculations, one can derive the following lemma

Lemma 2. *Let*

$$F(\rho) = \int_0^a f(r) J_\nu(\rho r) r dr.$$

Then F can be reconstructed from its samples via the formula

$$F(\rho) = \sum_{j=1}^{\infty} F(\alpha_{\nu,j}) \frac{2z_{\nu,j} J_\nu(a\rho)}{\left(z_{\nu,j}^2 - a^2\rho^2\right) J_{\nu+1}(z_{\nu,j})}. \tag{6}$$

Lemma 3. *Let $f(r,t)$ be a signal periodic with period T and highest frequency N/T, that is*

$$f(r,t) = \sum_{n=-N}^{N} c_n(r) e^{2\pi int/T}.$$

Then f can be reconstructed from $2N+1$ samples via

$$f(r,t) = \sum_{k=-N}^{N} f\left(r, \frac{kT}{2N+1}\right) \sigma_k(t), \tag{7}$$

where

$$\sigma_k(t) = \frac{\sin\left[(2N+1)\frac{\pi}{T}\left(t - \frac{kT}{2N+1}\right)\right]}{(2N+1)\sin\left[\frac{\pi}{T}(t - \frac{kT}{2N+1})\right]} \tag{8}$$

Proof. We have

$$f(r,t) = \sum_{n=-N}^{N} c_n(r) e^{2\pi int/T}, \tag{9}$$

and if we put $\eta = T/(2N+1)$, it follows that

$$\sum_{k=-N}^{N} f(r, k\eta)\, e^{-2\pi imk/(2N+1)} = \sum_{k,n=-N}^{N} c_n(r) e^{ik\tau}. \tag{10}$$

where $\tau = \frac{2\pi l}{2N+1}$ with $l = n - m$. The second summation on the right-hand side, i.e, the summation over k can be written in the form

$$\begin{aligned} \sum_{k=-N}^{N} e^{ik\tau} &= \frac{e^{-iN\tau}\left(1 - e^{(2N+1)i\tau}\right)}{1 - e^{i\tau}} \\ &= \frac{\sin \pi l}{\sin(\pi l/(2N+1))} = 0, \quad \text{if } l \neq 0, \end{aligned}$$

and when $l = 0$, i.e., $n = m$, we have $\tau = 0$ and $\sum_{k=-N}^{N} e^{i\tau k} = 2N+1$. By substituting this result into (10), we obtain

$$\sum_{k=-N}^{N} f(r, k\eta)\, e^{-2\pi imk/(2N+1)} = (2N+1)c_m.$$

Solving for c_n and substituting into Eq. (9), we obtain

$$f(r,t) = \frac{1}{2N+1} \sum_{k=-N}^{N} f(r, k\eta) \sum_{n=-N}^{N} e^{inx}, \tag{11}$$

where $x = \frac{2\pi}{T}\left(t - \frac{kT}{2N+1}\right)$. The second summation is easily seen to be $\sum_{n=-N}^{N} e^{inx} = \frac{\sin(N+1/2)x}{\sin(x/2)}$. Thus, Eq. (11) takes the desired form given by (7). □

Lemma 4. *Let*

$$\sigma_k(t) = \frac{\sin\left[(2N+1)\frac{\pi}{T}\left(t - \frac{Tk}{2N+1}\right)\right]}{(2N+1)\sin\left[\frac{\pi}{T}(t - \frac{Tk}{2N+1})\right]}; \quad -N \leq k \leq N.$$

Then

$$\int_0^T \sigma_k(t) e^{-i2\pi nt/T} dt = \begin{cases} \frac{T}{2N+1} e^{-2\pi ikn/(2N+1)} & -N \leq n \leq N \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Let $x_k = \frac{2\pi}{T}\left(t - \frac{kT}{2N+1}\right)$. The result will now follow from the observation that

$$\sigma_k(t) = \left(e^{-iNx_k} + \cdots + e^{iNx_k}\right)/(2N+1).$$

□

Now we are able to sketch the proof of the main theorem whose full proof will be published somewhere else.

Theorem 1. *Let f be bandlimited to a disc centered at the origin with radius R and with highest frequency $N/(2\pi)$, that is*

$$f(r,\theta)=\sum_{n=-N}^{N}c_n(r)e^{in\theta},\quad 0\le r\le R.$$

Let $F(\rho,\phi)$ be its canonical Fourier transform. Then F can be reconstructed from its samples according to the following formula

$$\tilde F(\rho,\phi)=\sum_{k,n=-N}^{N}\frac{e^{in(\phi-k\tau)}}{(2N+1)b}\sum_{j=1}^{\infty}\Phi_{n,j}(\rho/b)\tilde F(b\alpha_{n,j},\tau k),$$

where $\tilde F(\rho,\phi)=e^{-id\rho^2/2b}F(\rho,\phi)$,

$$\Phi_{n,j}(\rho/b)=\frac{2z_{n,j}J_n(R\rho/b)}{(z_{n,j}^2-R^2\rho^2/b^2)J_{n+1}(z_{n,j})},\qquad(12)$$

and $\tau=2\pi/(2N+1)$.

Proof. The canonical Fourier transform of $f(r,\theta)$ is given by

$$F(\rho,\phi)=\frac{1}{2\pi b}\int_{\mathbb{R}^2}f(r,\theta)K(r,\rho,\theta,\phi)rdrd\theta,$$

where $K(r,\rho,\theta,\phi)=\exp\left[\frac{i}{2b}\left(ar^2+d\rho^2-2r\rho\cos(\theta-\phi\right)\right]$. By setting

$$\tilde F(\rho,\phi)=e^{-id\rho^2/2b}F(\rho,\phi),\ \text{and}\ \tilde f(r,\theta)=e^{iar^2/2b}f(r,\theta),$$

and using Eq. (2), we have

$$\begin{aligned}\tilde F(\rho,\phi)&=\frac{1}{2\pi b}\int_0^{2\pi}\int_0^R\tilde f(r,\theta)\\&\times\sum_{n=-\infty}^{\infty}(-i)^nJ_n(r\rho/b)e^{-i(\theta-\phi)n}rdrd\theta\\&=\frac{1}{2\pi b}\sum_{n=-\infty}^{\infty}(-i)^ne^{in\phi}\sum_{m=-N}^{N}\int_0^{2\pi}e^{i\theta(m-n)}d\theta\\&\times\int_0^R c_m(r)e^{iar^2/2b}J_n(r\rho/b)rdr\\&=\frac{1}{b}\sum_{n=-N}^{N}e^{in\phi}\hat c_n(\rho),\end{aligned}\qquad(13)$$

$$\text{where }\hat c_n(\rho)=\int_0^R C_n(r)J_n(r\rho/b)rdr$$

which is the Hankel transform of $C_n(r)=(-i)^nc_n(r)e^{iar^2/2b}$ scaled by $1/b$. Therefore, from the sampling formula for the Hankel transform Eq. (6), we have

$$\hat c_n(\rho)=\sum_{j=1}^{\infty}\hat c_n(b\alpha_{n,j})\Phi_{n,j}(\rho/b).\qquad(14)$$

where $\alpha_{n,j}=z_{n,j}/R$ and $z_{n,j}$ is the j-th zero of the Bessel function $J_n(x)$ and $\Phi_{n,j}(\rho)$ is given by

$$\Phi_{n,j}(\rho)=\frac{2z_{n,j}J_n(R\rho)}{(z_{n,j}^2-R^2\rho^2)J_{n+1}(z_{n,j})}.\qquad(15)$$

Since

$$\tilde F(\rho,\phi)=\frac{1}{b}\sum_{n=-N}^{N}e^{in\phi}\hat c_n(\rho),$$

it follows from Lemma 3 , that

$$\tilde F(\rho,\phi)=\frac{1}{b}\sum_{k=-N}^{N}\tilde F(\rho,k\tau)\sigma_k(\phi),\qquad(16)$$

where $\sigma_k(\phi)$ is given by (8) and $\tau=2\pi/(2N+1)$. From (13), we have $\tilde F(\rho,k\tau)=\frac{1}{b}\sum_{k=-N}^{N}e^{ink\tau}\hat c_n(\rho)$, where

$$\hat c_n(\rho)=\frac{b}{2\pi}\int_0^{2\pi}\tilde F(\rho,\phi)e^{-in\phi}d\phi.$$

Hence, by Eq.(16), we have

$$\begin{aligned}\hat c_n(b\alpha_{n,j})&=\frac{1}{2\pi}\int_0^{2\pi}e^{-in\phi}\left(\sum_{k=-N}^{N}\tilde F(b\alpha_{n,j},\tau k)\sigma_k(\phi)\right)d\phi\\&=\frac{1}{2N+1}\sum_{k=-N}^{N}\tilde F(b\alpha_{n,j},\tau k)e^{-ikn\tau},\end{aligned}$$

where the last equation follows from Lemma 4 with $T=2\pi$. Thus, from Eq. (14), we have

$$\hat c_n(\rho)=\frac{1}{2N+1}\sum_{j=1}^{\infty}\Phi_{n,j}(\rho/b)\sum_{k=-N}^{N}\tilde F(b\alpha_{n,j},\tau k)e^{-ikn\tau}.\qquad(17)$$

Finally by substituting Eq. (17) into (13), we obtain the result ☐

Remark: By putting $a=\cos\theta=d, b=\sin\theta=-c$, in the above theorem, we obtain a sampling formula for signals that are bandlimited to a disc in the fractional Fourier transform.

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