Equi-isoclinic subspaces from difference sets

Matthew Fickus & Courtney A. Schmitt
Department of Mathematics and Statistics
Air Force Institute of Technology
Wright-Patterson Air Force Base, Ohio 45433
Email: Matthew.Fickus@afit.edu

***Abstract*—Equi-chordal tight fusion frames (ECTFFs) and equi-isoclinic tight fusion frames (EITFFs) are types of optimal packings in Grassmannian spaces. In particular, an ECTFF is an arrangement of equi-dimensional subspaces of a Euclidean space with the property that the smallest chordal distance between any pair of these subspaces is as large as possible. An EITFF is a special type of ECTFF that also happens to be an optimal packing with respect to the spectral distance. In the special case where the subspaces are one-dimensional, both ECTFFs and EITFFs reduce to optimal packings of lines known as equiangular tight frames (ETFs); these lines have minimal coherence, achieving equality in the Welch bound. ETFs are tricky to construct, but several infinite families of them are known. Harmonic ETFs in particular arise by restricting the characters of a finite abelian group to a difference set for that group. Moreover, there is a simple tensor-based method for combining ETFs with orthonormal bases to form EITFFs. It is an open question as to whether every EITFF essentially arises in this way.**

In this short paper, we preview a new result relating difference sets, harmonic ETFs, ECTFFs and EITFFs. This work expands on other recent results showing that certain harmonic ETFs are comprised of a number of simpler ETFs known as regular simplices; such ETFs arise, for example, from McFarland difference sets as well as the complements of certain Singer difference sets. It is already known that in this situation the subspaces spanned by these regular simplices necessarily form an ECTFF. We recently discovered that these same subspaces form an EITFF in some, but not all, cases. In an upcoming journal article, we shall characterize the properties of the underlying difference sets that lead to EITFFs in this fashion.

I. Equi-isoclinic tight fusion frames

Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$, and let $\mathbb{H}$ be a D-dimensional Hilbert space over $\mathbb{F}$ whose inner product is conjugate-linear in its first argument. Letting $\mathcal{N}$ be an N-element indexing set, a sequence $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ of M-dimensional subspaces of $\mathbb{H}$ is a *tight fusion frame* (TFF) for $\mathbb{H}$ if their orthogonal projection operators $\{\mathbf{P}_n\}_{n\in\mathcal{N}}$ satisfy $\sum_{n\in\mathcal{N}}\mathbf{P}_n=C\mathbf{I}$ for some $C>0$. TFFs are thus those sequences $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ of M-dimensional subspaces of $\mathbb{H}$ that achieve equality in the following application of the Cauchy-Schwarz inequality with respect to the Frobenius inner product $\langle\mathbf{A},\mathbf{B}\rangle_{\mathrm{Fro}}:=\mathrm{Tr}(\mathbf{A}^*\mathbf{B})$:

$$MN=\Bigl\langle\mathbf{I},\sum_{n\in\mathcal{N}}\mathbf{P}_n\Bigr\rangle_{\mathrm{Fro}}\leq\sqrt{D}\Bigl\|\sum_{n\in\mathcal{N}}\mathbf{P}_n\Bigr\|_{\mathrm{Fro}}.$$

Squaring this bound and simplifying the right-hand side gives

$$\tfrac{MN(MN-D)}{D}\leq\sum_{n\in\mathcal{N}}\sum_{n'\neq n}\langle\mathbf{P}_n,\mathbf{P}_{n'}\rangle_{\mathrm{Fro}}\qquad(1)$$

$$\leq N(N-1)\max_{n\neq n'}\langle\mathbf{P}_n,\mathbf{P}_{n'}\rangle_{\mathrm{Fro}},\qquad(2)$$

where (1) holds with equality if and only if $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is a TFF, while (2) holds with equality if and only if $\langle\mathbf{P}_n,\mathbf{P}_{n'}\rangle_{\mathrm{Fro}}$ is constant over all $n\neq n'$. When this second property holds, we say $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is *equi-chordal* since

$$\mathrm{dist}_{\mathrm{c}}^2(\mathcal{U}_n,\mathcal{U}_{n'})=\tfrac12\|\mathbf{P}_n-\mathbf{P}_{n'}\|_{\mathrm{Fro}}^2=M-\langle\mathbf{P}_n,\mathbf{P}_{n'}\rangle_{\mathrm{Fro}}\qquad(3)$$

is the *(squared) chordal distance* between $\mathcal{U}_n$ and $\mathcal{U}_{n'}$. Overall, for any N subspaces $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ of a D-dimensional Hilbert space, each of dimension M, we have

$$\tfrac{M(MN-D)}{D(N-1)}\leq\max_{n\neq n'}\langle\mathbf{P}_n,\mathbf{P}_{n'}\rangle_{\mathrm{Fro}},\qquad(4)$$

where equality holds if and only if $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is an *equi-chordal tight fusion frame* (ECTFF) for $\mathbb{H}$. Rewriting this inequality in terms of (3) gives an equivalent upper bound on $\max_{n\neq n'}\mathrm{dist}_{\mathrm{c}}(\mathcal{U}_n,\mathcal{U}_{n'})$, namely Conway, Hardin and Sloane's *simplex bound* [1]. By achieving this bound, ECTFFs are optimal packings of N points—with respect to the chordal distance—in the *Grassmannian space* that consists of M-dimensional subspaces of $\mathbb{H}$.

To proceed, it helps to introduce some operator notation. For any finite indexing set $\mathcal{N}$, let $\mathbb{F}^{\mathcal{N}}:=\{\mathbf{x}:\mathcal{N}\to\mathbb{F}\}$. The *synthesis operator* of any sequence of vectors $\{\mathbf{a}_n\}_{n\in\mathcal{N}}$ in $\mathbb{H}$ is $\mathbf{A}:\mathbb{F}^{\mathcal{N}}\to\mathbb{H}$, $\mathbf{A}\mathbf{x}:=\sum_{n\in\mathcal{N}}\mathbf{x}(n)\mathbf{a}_n$. Its adjoint is the corresponding *analysis operator* $\mathbf{A}^*:\mathbb{H}\to\mathbb{F}^{\mathcal{N}}$, which is given by $(\mathbf{A}^*\mathbf{y})(n)=\langle\mathbf{a}_n,\mathbf{y}\rangle$. Composing these operators gives $\mathbf{A}^*\mathbf{A}:\mathbb{F}^{\mathcal{N}}\to\mathbb{F}^{\mathcal{N}}$, which is naturally regarded as an $(\mathcal{N}\times\mathcal{N})$-indexed *Gram* matrix whose (n,n')th entry is $(\mathbf{A}^*\mathbf{A})(n,n')=\langle\mathbf{a}_n,\mathbf{a}_{n'}\rangle$, as well as the corresponding *frame operator* $\mathbf{A}\mathbf{A}^*:\mathbb{H}\to\mathbb{H}$, $\mathbf{A}\mathbf{A}^*\mathbf{y}=\sum_{n\in\mathcal{N}}\langle\mathbf{a}_n,\mathbf{y}\rangle\mathbf{a}_n$. Regarding any single vector $\mathbf{a}_n$ as a trivial synthesis operator defined by $\mathbf{a}_n(x):=x\mathbf{a}_n$ for any $x\in\mathbb{F}$, its adjoint is the linear functional $\mathbf{a}_n^*:\mathbb{H}\to\mathbb{F}$, $\mathbf{a}_n^*\mathbf{y}=\langle\mathbf{a}_n,\mathbf{y}\rangle$. Under this notation, $\mathbf{A}\mathbf{A}^*=\sum_{n\in\mathcal{N}}\mathbf{a}_n\mathbf{a}_n^*$.

If $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is any sequence of M-dimensional subspaces of $\mathbb{H}$, then letting $\mathcal{M}$ be some M-element indexing set, we, for each n, can let $\{\mathbf{e}_{n,m}\}_{m\in\mathcal{M}}$ be an orthonormal basis for $\mathcal{U}_n$, let $\mathbf{E}_n$ be its corresponding synthesis operator, and write the orthogonal projection operator onto $\mathcal{U}_n$ as $\mathbf{P}_n=\mathbf{E}_n\mathbf{E}_n^*$. As such,

$$\langle\mathbf{P}_n,\mathbf{P}_{n'}\rangle_{\mathrm{Fro}}=\mathrm{Tr}(\mathbf{E}_n\mathbf{E}_n^*\mathbf{E}_{n'}\mathbf{E}_{n'}^*)=\|\mathbf{E}_n^*\mathbf{E}_{n'}\|_{\mathrm{Fro}}^2.\qquad(5)$$

Here, $\mathbf{E}_n^*\mathbf{E}_{n'}$ is an $\mathcal{M}\times\mathcal{M}$ *cross-Gram* matrix whose (m,m')th entry is $\langle\mathbf{e}_{n,m},\mathbf{e}_{n',m'}\rangle$. Though these cross-Gram matrices are bases dependent, their singular values

$\{\sigma_{n,n',m}\}_{m=1}^{M}$ are not, and lie in the interval $[0,1]$: letting $\|\mathbf{A}\|_2$ denote the induced 2-norm of an operator $\mathbf{A}$, and writing singular values in decreasing order, we have $\sigma_{n,n',1} = \|\mathbf{E}_n^*\mathbf{E}_{n'}\|_2 \leq \|\mathbf{E}_n\|_2\|\mathbf{E}_{n'}\|_2 = 1$. Thus, there is an increasing sequence of angles $\{\theta_{n,n',m}\}_{m=1}^{M}$ in $[0,\frac{\pi}{2}]$ such that $\sigma_{n,n',m} = \cos(\theta_m)$ for all $n,n' \in \mathcal{N}$, $m = 1,\dots,M$; these are known as the *principal angles* between $\mathcal{U}_n$ and $\mathcal{U}_{n'}$. In particular, combining this with (3) and (5) gives $\operatorname{dist}_{c^2}(\mathcal{U}_n,\mathcal{U}_{n'}) = \sum_{m=1}^{M}\sin^2(\theta_{n,n',m})$. In contrast, the *(squared) spectral distance* [3] between $\mathcal{U}_n$ and $\mathcal{U}_{n'}$ is:

$$\operatorname{dist}_s^2(\mathcal{U}_n,\mathcal{U}_{n'}) = \sin^2(\theta_{n,n',1}) = 1 - \|\mathbf{E}_n^*\mathbf{E}_{n'}\|_2^2. \quad (6)$$

To obtain a pairwise spectral distance bound that is analogous to the simplex bound, note $\|\mathbf{E}_n^*\mathbf{E}_{n'}\|_{\mathrm{Fro}}^2 \leq M\|\mathbf{E}_n^*\mathbf{E}_{n'}\|_2^2$, where equality holds if and only if the principal angles $\{\theta_{n,n',m}\}_{m=1}^{M}$ are constant over m; when this occurs, $\mathcal{U}_n$ and $\mathcal{U}_{n'}$ are said to be *isoclinic*. This happens if and only if there exists some $\sigma_{n,n'} \geq 0$ such that $\mathbf{E}_n^*\mathbf{E}_{n'}\mathbf{E}_{n'}^*\mathbf{E}_n = \sigma_{n,n'}^2\mathbf{I}$, or equivalently, $\mathbf{P}_n\mathbf{P}_{n'}\mathbf{P}_n = \sigma_{n,n'}^2\mathbf{P}_n$.

Combining these facts with (4) and (5) then gives

$$\left[\tfrac{MN-D}{D(N-1)}\right]^{\frac{1}{2}} \leq \max_{n\neq n'}\|\mathbf{E}_n^*\mathbf{E}_{n'}\|_2 \quad (7)$$

where equality holds if and only if $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is an *equi-isoclinic tight fusion frame (EITFF)* for $\mathbb{H}$, namely an ECTFF where any pair of subspaces are isoclinic. That is, $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ achieves equality in (7) if and only if $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is a TFF whose principal angles $\theta_{n,n',m}$ are constant over all $n \neq n'$ and m, or equivalently, when there exists $\sigma \geq 0$ such that $\mathbf{P}_n\mathbf{P}_{n'}\mathbf{P}_n = \sigma^2\mathbf{P}_n$ for all $n \neq n'$. By rewriting (7) in terms of (6), we see that EITFFs are optimal packings in Grassmannian space with respect to the spectral distance.

In the special case where $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is any sequence of subspaces of $\mathbb{H}$ of dimension $M = 1$, any unit norm vector $\boldsymbol{\varphi}_n$ in $\mathcal{U}_n$ is an orthonormal basis for it, giving $\mathbf{E}_n = \boldsymbol{\varphi}_n$ and $\mathbf{P}_n = \boldsymbol{\varphi}_n\boldsymbol{\varphi}_n^*$. Here, $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is a TFF if and only if $C\mathbf{I} = \sum_{n\in\mathcal{N}}\mathbf{P}_n = \sum_{n\in\mathcal{N}}\boldsymbol{\varphi}_n\boldsymbol{\varphi}_n^* = \boldsymbol{\Phi}\boldsymbol{\Phi}^*$ for some $C > 0$, namely if and only if $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is a *tight frame* for $\mathbb{H}$. Moreover, for any $n \neq n'$, the cross-Gram matrix $\mathbf{E}_n^*\mathbf{E}_{n'}$ of $\mathcal{U}_n$ and $\mathcal{U}_{n'}$ is simply the scalar $\langle\boldsymbol{\varphi}_n,\boldsymbol{\varphi}_{n'}\rangle$, giving $\langle\mathbf{P}_n,\mathbf{P}_{n'}\rangle_{\mathrm{Fro}} = \|\mathbf{E}_n^*\mathbf{E}_{n'}\|_{\mathrm{Fro}}^2 = |\langle\boldsymbol{\varphi}_n,\boldsymbol{\varphi}_{n'}\rangle|^2 = \|\mathbf{E}_n^*\mathbf{E}_{n'}\|_2^2$. Thus, when $M = 1$, $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is equi-chordal precisely when it is equi-isoclinic, and this occurs if and only if $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is *equiangular*, namely when $|\langle\boldsymbol{\varphi}_n,\boldsymbol{\varphi}_{n'}\rangle|$ is constant over all $n \neq n'$. Moreover, when $M = 1$, both (4) and (7) reduce to the *Welch bound* [13]: for any unit norm vectors $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$,

$$\left[\tfrac{N-D}{D(N-1)}\right]^{\frac{1}{2}} \leq \max_{n\neq n'}|\langle\boldsymbol{\varphi}_n,\boldsymbol{\varphi}_{n'}\rangle|, \quad (8)$$

where equality is achieved if and only if $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is a tight frame for $\mathbb{H}$ that is also equiangular, namely an *equiangular tight frame* (ETF) for $\mathbb{H}$.

ECTFFs, EITFFs and ETFs are tricky to construct from scratch. That said, a number of infinite families of ETFs are now known [6]. Moreover, every ETF leads to an infinite family of EITFFs in a trivial way. To elaborate, let $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ be an ETF for a D-dimensional Hilbert space $\mathbb{H}$. For any M, let $\{\mathbf{u}_m\}_{m\in\mathcal{M}}$ be an orthonormal basis for an M-dimensional Hilbert space $\mathbb{K}$, meaning its synthesis operator $\mathbf{U}$ is unitary. For each $n \in \mathcal{N}$, define $\mathbf{e}_{n,m} := \boldsymbol{\varphi}_n \otimes \mathbf{u}_m \in \mathbb{H}\otimes\mathbb{K}$ for all $m \in \mathcal{M}$, and so the synthesis operator of $\{\mathbf{e}_{n,m}\}_{m\in\mathcal{M}}$ is $\mathbf{E}_n = \boldsymbol{\varphi}_n \otimes \mathbf{U}$. Since $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is a tight frame, $\mathcal{U}_n := \operatorname{span}\{\mathbf{e}_{n,m}\}_{m\in\mathcal{M}} = \operatorname{range}(\mathbf{E}_n)$ is a TFF: since $\mathbf{P}_n = \mathbf{E}_n\mathbf{E}_n^* = \boldsymbol{\varphi}_n\boldsymbol{\varphi}_n^* \otimes \mathbf{U}\mathbf{U}^* = \boldsymbol{\varphi}_n\boldsymbol{\varphi}_n^* \otimes \mathbf{I}$ for all n, $\sum_{n\in\mathcal{N}}\mathbf{P}_n = (\sum_{n\in\mathcal{N}}\boldsymbol{\varphi}_n\boldsymbol{\varphi}_n^*) \otimes \mathbf{I} = C(\mathbf{I}\otimes\mathbf{I})$ for some $C > 0$. Moreover, the corresponding cross-Gram matrices are $\mathbf{E}_n^*\mathbf{E}_{n'} = \boldsymbol{\varphi}_n^*\boldsymbol{\varphi}_{n'} \otimes \mathbf{U}^*\mathbf{U} = \langle\boldsymbol{\varphi}_n,\boldsymbol{\varphi}_{n'}\rangle \otimes \mathbf{I}$. In particular, for every n we have $\mathbf{E}_n^*\mathbf{E}_n = \mathbf{I}$ and so $\{\mathbf{e}_{n,m}\}_{m\in\mathcal{M}}$ is an orthonormal basis for $\mathcal{U}_n$. Moreover, for any $n \neq n'$, every singular value of $\mathbf{E}_n^*\mathbf{E}_{n'}$ is equal to $|\langle\boldsymbol{\varphi}_n,\boldsymbol{\varphi}_{n'}\rangle| = [\frac{N-D}{D(N-1)}]^{\frac{1}{2}}$. As such, $\{\mathcal{U}_n\}_{n\in\mathcal{N}}$ is an EITFF for the MD-dimensional space $\mathbb{H}\otimes\mathbb{K}$ that consists of N subspaces of dimension M. Though not every known EITFF is quite this simple, the D, M and N parameters of every known EITFF can be obtained from the parameters of a known ETF in this way. This leads to the following conjecture [5]:

Conjecture 1: If there exists an EITFF for a D-dimensional Hilbert space that consists of N subspaces of dimension M, then M divides D and there exists an N-vector ETF for a Hilbert space of dimension $\frac{D}{M}$.

In a recent paper [4], a new method to produce ECTFFs from certain ETFs was discovered. In an upcoming journal article [8], we show that in some cases, these ECTFFs are actually EITFFs. Though these results neither prove nor disprove the above conjecture in general, they do prove that it at least holds in a special case. In the remainder of this document, we outline this result.

II. Harmonic ETFs and difference sets

A *character* on a finite abelian group $\mathcal{G}$ is a homomorphism $\gamma : \mathcal{G} \to \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The *(Pontryagin) dual* of $\mathcal{G}$ is the set $\hat{\mathcal{G}}$ of all characters of $\mathcal{G}$, which is itself a group under pointwise multiplication. It is well known that $\hat{\mathcal{G}}$ is isomorphic to $\mathcal{G}$, and that $\{\gamma\}_{\gamma\in\hat{\mathcal{G}}}$ is an equal-norm orthogonal basis for $\mathbb{C}^{\mathcal{G}}$. As such, the synthesis operator of $\{\gamma\}_{\gamma\in\hat{\mathcal{G}}}$, namely the $(\mathcal{G}\times\hat{\mathcal{G}})$-indexed *character table* $\mathbf{F}$ defined by $\mathbf{F}(g,\gamma) := \gamma(g)$, has the property that $\mathbf{F}^{-1} = \frac{1}{G}\mathbf{F}^*$, where G is the order of $\mathcal{G}$. Here, the corresponding analysis operator is the *discrete Fourier transform* (DFT) over $\mathcal{G}$ defined by $(\mathbf{F}^*\mathbf{y})(\gamma) := \langle\gamma,\mathbf{y}\rangle$ for any $\mathbf{y} \in \mathbb{C}^{\mathcal{G}}$ and $\gamma \in \hat{\mathcal{G}}$.

Since $\mathbf{F}\mathbf{F}^* = G\mathbf{I}$, the rows of $\mathbf{F}$ are equal-norm orthogonal. Of course, any subset of these rows also has this property: if $\mathcal{D}$ is any nonempty D-element subset of $\mathcal{G}$, then letting $\boldsymbol{\Phi}$ be the $(\mathcal{D}\times\hat{\mathcal{G}})$-index submatrix defined by $\boldsymbol{\Phi}(d,\gamma) = D^{-\frac{1}{2}}\gamma(d)$, we have $\boldsymbol{\Phi}\boldsymbol{\Phi}^* = \frac{G}{D}\mathbf{I}$. Regarding the γ-indexed column of $\boldsymbol{\Phi}$ as the vector $\boldsymbol{\varphi}_\gamma = D^{-\frac{1}{2}}\gamma \in \mathbb{C}^{\mathcal{D}}$, we equivalently have that $\{\boldsymbol{\varphi}_\gamma\}_{\gamma\in\hat{\mathcal{G}}}$ is a tight frame for $\mathbb{C}^{\mathcal{D}}$. Such tight frames are dubbed *harmonic frames*, and have a circulant Gram matrix with $\langle\boldsymbol{\varphi}_\gamma,\boldsymbol{\varphi}_{\gamma'}\rangle = \frac{1}{D}\sum_{d\in\mathcal{D}}(\gamma^{-1}\gamma')(d) = \frac{1}{D}(\mathbf{F}^*\boldsymbol{\chi}_{\mathcal{D}})(\gamma(\gamma')^{-1})$ where $\boldsymbol{\chi}_{\mathcal{D}}$ is the characteristic function of $\mathcal{D}$. In particular, a harmonic frame is an ETF if and only if $|(\mathbf{F}^*\boldsymbol{\chi}_{\mathcal{D}})(\gamma)|^2$ is constant over all $\gamma \neq 1$. To continue, we exploit the fact that the *convolution* $(\mathbf{y}_1 * \mathbf{y}_2)(g) := \sum_{g'\in\mathcal{G}} \mathbf{y}_1(g')\mathbf{y}_2(g-g')$ of $\mathbf{y}_1$ and $\mathbf{y}_2$ satisfies $\mathbf{F}^*(\mathbf{y}_1 * \mathbf{y}_2) = (\mathbf{F}^*\mathbf{y}_1)(\mathbf{F}^*\mathbf{y}_2)$, while the *involution* $\tilde{\mathbf{y}}(g) := \overline{\mathbf{y}(-g)}$ satisfies $\mathbf{F}^*\tilde{\mathbf{y}} = \overline{\mathbf{F}^*\mathbf{y}}$. As such, $|\mathbf{F}^*\boldsymbol{\chi}_\mathcal{D}|^2 = \mathbf{F}^*(\boldsymbol{\chi}_\mathcal{D} * \tilde{\boldsymbol{\chi}}_\mathcal{D})$ where $\boldsymbol{\chi}_\mathcal{D} * \tilde{\boldsymbol{\chi}}_\mathcal{D}$ is the *autocorrelation* of $\boldsymbol{\chi}_\mathcal{D}$, given by:

$$\begin{aligned}(\boldsymbol{\chi}_\mathcal{D} * \tilde{\boldsymbol{\chi}}_\mathcal{D})(g) &= \sum_{g'\in\mathcal{D}} \boldsymbol{\chi}_\mathcal{D}(g')\boldsymbol{\chi}_\mathcal{D}(g'-g)\\ &= \#[D\cap(g+D)]\\ &= \#\{(d,d')\in\mathcal{D}\times\mathcal{D} : g = d-d'\}. \end{aligned} \tag{9}$$

We also observe $(\mathbf{F}^*\mathbf{y})(\gamma)$ is constant over all $\gamma \neq 1$ if and only if $\mathbf{y}(g)$ is constant over all $g \neq 0$. Altogether, these facts imply the following result, a fact observed in [12], and later rediscovered in the context of ETFs [11], [14], [2]: the harmonic frame arising from a subset $\mathcal{D}$ of $\mathcal{G}$ is an ETF if and only if $\mathcal{D}$ is a *difference set* of $\mathcal{G}$, namely if and only if the number $(\boldsymbol{\chi}_\mathcal{D} * \tilde{\boldsymbol{\chi}}_\mathcal{D})(g)$ of ways a given $g \in \mathcal{G}$ can be written as a difference of members of $\mathcal{D}$ is constant over all $g \neq 0$.

For a simple example of a difference set, for any finite abelian group $\mathcal{G}$, the set $\mathcal{D} = \mathcal{G}\backslash\{0\}$ is a difference set for $\mathcal{G}$, and yields a harmonic ETF for a space of dimension $D = G-1$ that consists of $N = G$ vectors; in general, any ETF with $N = D+1$ is known as a *regular D-simplex*. More sophisticated constructions lead to harmonic ETFs with a wide variety of (D, N) parameters, including *Singer* and *McFarland* difference sets, both of which arise from hyperplanes in vector spaces over finite fields [10].

To elaborate, for any prime power Q and any integer $J \geq 2$, the finite field $\mathbb{F}_{Q^J}$ of order Q^J is a J-dimensional vector space over its subfield $\mathbb{F}_Q$. The *field trace* $\mathrm{tr} : \mathbb{F}_{Q^J} \to \mathbb{F}_Q$, $\mathrm{tr}(\beta) := \sum_{j=0}^{J-1} \beta^{Q^j}$ is a nontrivial linear functional, and so its kernel $\mathcal{U} := \{\beta \in \mathbb{F}_{Q^J} : \mathrm{tr}(\beta) = 0\}$ is a hyperplane, that is, has codimension 1. In fact, every linear functional on $\mathbb{F}_{Q^J}$ is of the form $\beta \mapsto \mathrm{tr}(\gamma\beta)$ for some $\gamma \neq 0$; this follows from the decomposition $\beta = \sum_{j=1}^{J} \mathrm{tr}(\gamma_j\beta)\delta_j$, where $\{\gamma_j\}_{j=1}^J$ and $\{\delta_j\}_{j=1}^J$ are bases for $\mathbb{F}_{Q^J}$ that are biorthogonal with respect to the "dot product" $\gamma\cdot\delta := \mathrm{tr}(\gamma\delta)$. As such, every hyperplane in $\mathbb{F}_{Q^J}$ is of the form $\gamma\mathcal{U} = \{\beta \in \mathbb{F}_{Q^J} : \mathrm{tr}(\gamma^{-1}\beta) = 0\}$ for some $\gamma \neq 0$. Moreover, since $\gamma\mathcal{U} = \mathcal{U}$ if and only if γ lies in the multiplicative group $\mathbb{F}_Q^\times$ of $\mathbb{F}_Q$, and since any two distinct hyperplanes intersect in a subspace of codimension 2,

$$\#[\mathcal{U}\cap(\gamma\mathcal{U})] = \begin{cases} Q^{J-1}, & \gamma \in \mathbb{F}_Q^\times,\\ Q^{J-2}, & \gamma \in \mathbb{F}_{Q^J}^\times\backslash\mathbb{F}_Q^\times. \end{cases} \tag{10}$$

Comparing against (9), we see that the hyperplane $\mathcal{U}$ is almost, but not quite, a difference set. To finalize the construction, we *projectivize*: any nonzero β in $\mathbb{F}_{Q^J}$ is contained in a unique 1-dimensional subspace of $\mathbb{F}_Q$, as indicated by the quotient map from $\mathbb{F}_{Q^J}^\times \to \mathbb{F}_{Q^J}^\times/\mathbb{F}_Q^\times$, $\beta \mapsto \overline{\beta} := \beta\mathbb{F}_Q^\times$. In particular, projectivizing $\mathcal{U}$ yields

$$\mathcal{D} = \{\overline{\beta} \in \mathbb{F}_{Q^J}^\times/\mathbb{F}_Q^\times : \mathrm{tr}(\beta) = 0\}. \tag{11}$$

Since projectivizing identifies any nonzero $\beta \in \mathbb{F}_{Q^J}$ with $Q-2$ other such vectors, (10) becomes

$$\#[\mathcal{D}\cap(\overline{\gamma}\mathcal{D})] = \tfrac{1}{Q-1}\begin{cases} Q^{J-1}-1, & \overline{\gamma} = \overline{1},\\ Q^{J-2}-1, & \overline{\gamma} \neq \overline{1}, \end{cases}$$

and so $\mathcal{D}$ is a difference set for $\mathcal{G} = \mathbb{F}_{Q^J}^\times/\mathbb{F}_Q^\times$. To make this construction more explicit, recall that the multiplicative group of any finite field is cyclic. In particular, $\mathbb{F}_{Q^J}^\times = \langle\alpha\rangle \cong \mathbb{Z}_{Q^J-1}$ where α is the root of a primitive polynomial over $\mathbb{F}_Q$ of degree J. Here, $\mathbb{F}_Q^\times$ is isomorphic to the subgroup of $\mathbb{Z}_{Q^J-1}$ of order $Q-1$. As such, abusing notation, we may equivalently regard our difference set $\mathcal{D}$ and group $\mathcal{G}$ as

$$\mathcal{D} = \{k = 0, \dots \tfrac{Q^J-1}{Q-1} - 1 : \mathrm{tr}(\alpha^k) = 0\}, \quad \mathcal{G} = \mathbb{Z}_{\frac{Q^J-1}{Q-1}}.$$

Such $\mathcal{D}$ are known as Singer difference sets, and yield harmonic ETFs consisting of $G = \frac{Q^J-1}{Q-1}$ vectors in a space of dimension $D = \frac{Q^{J-1}-1}{Q-1}$.

III. ETFs comprised of regular simplices

If $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ is an N-vector ETF for a D-dimensional space $\mathbb{H}$, a subsequence of these vectors is still equiangular, and, in rare cases, can itself form an ETF for a subspace of $\mathbb{H}$. In fact, some ETFs have the remarkable property that their vectors can be partitioned into subsequences that are regular S-simplices for their spans; when this occurs, we say the ETF is *comprised of regular simplices*. Here, since the Welch bound (8) of $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}}$ equals that of a simplex it contains,

$$\big[\tfrac{N-D}{D(N-1)}\big]^{\frac12} = \big\{\tfrac{(S+1)-S}{S[(S+1)-1]}\big\}^{\frac12} = \tfrac{1}{S}, \tag{12}$$

and so S is necessarily the reciprocal of the Welch bound (8).

For example, a *Steiner ETF* is formed by using a *balanced incomplete block design* to embed S-simplices into subspaces of $\mathbb{H}$ in a way that permits the entire ensemble to be equiangular [7]; such ETFs are thus comprised of regular simplices by design. Moreover, harmonic ETFs arising from McFarland difference sets are unitarily equivalent to a special class of Steiner ETFs [9], and so also have this property. Recently, it has been shown that certain other harmonic ETFs—ones that are provably not unitarily equivalent to any Steiner ETF—are nevertheless comprised of regular simplices [4]. Here, the main idea is to exploit the *Poisson summation formula*: if $\mathcal{H}$ is any H-element subgroup of $\mathcal{G}$, then $\mathbf{F}^*\boldsymbol{\chi}_\mathcal{H} = H\boldsymbol{\chi}_{\mathcal{H}^\perp}$ where $\mathcal{H}^\perp := \{\gamma \in \hat{\mathcal{G}} : \gamma(h) = 1,\ \forall h \in \mathcal{H}\}$ is the *annihilator* of $\mathcal{H}$. It is well known that $\mathcal{H}^\perp$ is itself a subgroup of $\hat{\mathcal{G}}$ of order $\frac{G}{H}$, and in fact is isomorphic to the dual of $\mathcal{G}/\mathcal{H}$.

To elaborate, for any subgroup $\mathcal{H}$ of $\mathcal{G}$ that is disjoint from a difference set $\mathcal{D}$, the corresponding harmonic ETF $\{\boldsymbol{\varphi}_\gamma\}_{\gamma\in\hat{\mathcal{G}}}$ can be partitioned according to the cosets of $\mathcal{H}^\perp$, where vectors from any coset sum to zero: for any $\gamma \in \hat{\mathcal{G}}$,

$$\sum_{\gamma'\in\gamma\mathcal{H}^\perp} \boldsymbol{\varphi}_{\gamma'}(d) = \tfrac{\gamma(d)}{\sqrt{D}}(\mathbf{F}\boldsymbol{\chi}_{\mathcal{H}^\perp})(d) = \tfrac{G}{H\sqrt{D}}\boldsymbol{\chi}_\mathcal{H}(d) = 0.$$

In particular, if $\mathcal{D}$ is disjoint from a subgroup $\mathcal{H}$ of order $H = \frac{G}{S+1}$ where S satisfies (12) with $N = G$, then every coset-indexed subsequence $\{\boldsymbol{\varphi}_{\gamma'}\}_{\gamma'\in\gamma\mathcal{H}^\perp}$ of $\{\boldsymbol{\varphi}_\gamma\}_{\gamma\in\hat{\mathcal{G}}}$ is a regular simplex for its span $\mathcal{U}_{\overline{\gamma}}$, being a sequence of $S+1$ vectors that lie in an S-dimensional subspace of $\mathbb{H}$, while achieving the corresponding Welch bound. In [4], it is further observed that the subspaces $\{\mathcal{U}_{\overline{\gamma}} : \overline{\gamma} \in \hat{\mathcal{G}}/\mathcal{H}^\perp\}$ necessarily form an ECTFF for $\mathbb{C}^\mathcal{D}$: for any $\gamma \in \hat{\mathcal{G}}$, the orthogonal projection operator onto

$\mathcal{U}_{\overline{\gamma}}$ is $\mathbf{P}_{\overline{\gamma}} = \frac{S}{S+1}\mathbf{\Phi}_\gamma\mathbf{\Phi}_\gamma^*$ where $\mathbf{\Phi}_\gamma$ is the synthesis operator for $\{\boldsymbol{\varphi}_{\gamma'}\}_{\gamma'\in\gamma\mathcal{H}^\perp}$, and so $\sum_{\overline{\gamma}\in\hat{\mathcal{G}}/\mathcal{H}^\perp}\mathbf{P}_{\overline{\gamma}} = \frac{S}{S+1}\mathbf{\Phi}\mathbf{\Phi}^* = \frac{GS}{D(S+1)}\mathbf{I}$; moreover, for any $\gamma,\gamma'\in\hat{\mathcal{G}}$ such that $\gamma\mathcal{H}^\perp\neq\gamma'\mathcal{H}^\perp$,

$$\langle\mathbf{P}_\gamma,\mathbf{P}_{\gamma'}\rangle = \tfrac{S^2}{(S+1)^2}\|\mathbf{\Phi}_\gamma^*\mathbf{\Phi}_{\gamma'}\|_{\mathrm{Fro}}^2 = 1,$$

since $\mathbf{\Phi}_\gamma^*\mathbf{\Phi}_{\gamma'}$ is an $(S+1)\times(S+1)$ matrix, each of whose entries have modulus $\frac{1}{S}$. In general, ECTFFs that arise in this way are not necessarily EITFFs. For example, harmonic ETFs arising from McFarland difference sets are comprised of regular simplices in this way [4], and yet are unitarily equivalent to certain Steiner ETFs, implying the rank of the cross-Gram matrices $\mathbf{E}_\gamma^*\mathbf{E}_{\gamma'}$ is 1, not S. That said, in an upcoming paper, we prove that in some cases, the ECTFFs that arise from harmonic ETFs in this fashion are EITFFs:

Theorem 1 ([8]): Let $\mathcal{D}$ be a D-element difference set for an abelian group $\mathcal{G}$ of order G, and let $\mathcal{D}$ be disjoint from a subgroup $\mathcal{H}$ of $\mathcal{G}$ of order $H = \frac{G}{S+1}$ where $S = [\frac{G(D-1)}{G-D}]^{\frac12}$. Let $\{\mathcal{U}_{\overline{\gamma}}\}_{\overline{\gamma}\in\hat{\mathcal{G}}/\mathcal{H}^\perp}$ be the sequence of S-dimensional subspaces of $\mathbb{C}^{\mathcal{D}}$ spanned by the regular simplices that comprise the corresponding harmonic ETF. Then, letting $\mathcal{D}_g := (\mathcal{D}-g)\cap\mathcal{H}$ for any $g\in\mathcal{G}$, we have $\#(\mathcal{D}_g) = \frac{D}{S}$ for any $g\notin\mathcal{H}$, and moreover, $\{\mathcal{U}_{\overline{\gamma}}\}_{\overline{\gamma}\in\hat{\mathcal{G}}/\mathcal{H}^\perp}$ is an EITFF for $\mathbb{C}^{\mathcal{D}}$ if and only if each $\mathcal{D}_g$ is a difference set for $\mathcal{H}$.

Essentially, this result states that when a harmonic ETF forms an EITFF in this way, the underlying difference set $\mathcal{D}$ for $\mathcal{G}$ is comprised of S difference sets for $\mathcal{H}$, each of cardinality $\frac{D}{S}$. In particular, Conjecture 1 holds for such EITFFs.

We conclude by discussing how Theorem 1 is not vacuous. In fact, there is an infinite family of difference sets that yield EITFFs in this way. To elaborate, for any prime power Q and integer $J\geq 3$, the complement of the Singer difference set (11) is itself a (cyclic) difference set with parameters

$$D = Q^{J-1},\quad G = \tfrac{Q^J-1}{Q-1},\quad S = \bigl[\tfrac{D(G-1)}{G-D}\bigr]^{\frac12} = Q^{\frac{J}{2}},$$

and so the corresponding harmonic ETF is comprised of regular simplices if it is disjoint from a subgroup $\mathcal{H}$ of order $H = \frac{G}{S+1} = (Q^{\frac{J}{2}}-1)/(Q-1)$. Since $\mathcal{G}$ is cyclic, such a subgroup $\mathcal{H}$ is necessarily unique, and moreover exists if and only if $Q^{\frac{J}{2}}$ is an integer, that is, when either Q is a perfect square or J is even. In fact, when J is even, $\mathbb{F}_{Q^{J/2}}$ is a subfield of $\mathbb{F}_{Q^J}$, implying $\mathcal{H} = \mathbb{F}_{Q^{J/2}}^\times/\mathbb{F}_Q^\times$ is the unique subgroup of $\mathcal{G} = \mathbb{F}_{Q^J}^\times/\mathbb{F}_Q^\times$ of the appropriate order.

Summarizing, for any prime power Q and even integer $J\geq 4$, the harmonic ETF arising from the complement of a Singer difference set (11) for $\mathcal{G} = \mathbb{F}_{Q^J}^\times/\mathbb{F}_Q^\times$ is comprised of regular simplices if it is disjoint from $\mathcal{H} = \mathbb{F}_{Q^{J/2}}^\times/\mathbb{F}_Q^\times$. Equivalently, the Singer difference set (11) must itself contain $\mathcal{H}$. Remarkably, while the Singer difference set might not have this property, a shift of it—an alternative difference set of the same cardinality—always does. Indeed, when J is even, the field trace from $\mathbb{F}_{Q^J}$ to $\mathbb{F}_Q$ factors through the field traces to and from $\mathbb{F}_{Q^{J/2}}$: the "freshman's dream" implies $\mathrm{tr}(\beta) = \sum_{j=0}^{J-1}\beta^{Q^j} = \sum_{j=0}^{J/2-1}(\beta+\beta^{Q^{J/2}})^{Q^j}$. In particular, if $\beta^{Q^{J/2}-1} = -1$ then $\mathrm{tr}(\beta) = 0$ and $\overline{\beta}$ lies in the Singer difference set (11). At the same time, $\beta^{Q^{J/2}-1} = 1$ for any $\beta\in\mathbb{F}_{Q^{J/2}}$. As such, in the special case where Q is even, $1=-1$ and so the Singer difference set (11) indeed contains $\mathcal{H} = \mathbb{F}_{Q^{J/2}}^\times/\mathbb{F}_Q^\times$. If instead Q is odd, $-1 = \alpha^{(Q^J-1)/2}$ where α is the generator of $\mathbb{F}_{Q^J}^\times$. This implies $\beta_0 := \alpha^{(Q^{J/2}+1)/2}$ satisfies $\beta_0^{Q^{J/2}-1} = -1$, and so $\mathrm{tr}(\beta_0\beta) = 0$ for all $\beta\in\mathbb{F}_{Q^{J/2}}^\times$. As such, when Q is odd, $\mathcal{H}$ is contained in the shift of the Singer difference set (11) by $\overline{\beta_0}^{-1}$.

Altogether, we see that regardless of whether or not Q is even, when $J\geq 4$ is even, there is always some translation of the Singer difference set (11) for $\mathcal{G} = \mathbb{F}_{Q^J}^\times/\mathbb{F}_Q^\times$ so that its complement $\mathcal{D}$ is disjoint from the subgroup $\mathcal{H} = \mathbb{F}_{Q^{J/2}}^\times/\mathbb{F}_Q^\times$ of order $H = \frac{G}{S+1}$. As such, the corresponding harmonic ETF yields an EITFF in the manner of Theorem 1 if and only if $\mathcal{D}_g := (\mathcal{D}-g)\cap\mathcal{H}$ is a difference set for $\mathcal{H}$ for all $g\in\mathcal{G}$. Taking complements, this equates to having

$$\begin{aligned}&\overline{\gamma}^{-1}\{\overline{\beta}\in\mathbb{F}_{Q^J}^\times/\mathbb{F}_Q^\times : \mathrm{tr}(\beta)=0\}\cap(\mathbb{F}_{Q^{J/2}}^\times/\mathbb{F}_Q^\times)\\ &\qquad = \{\overline{\beta}\in\mathbb{F}_{Q^{J/2}}^\times/\mathbb{F}_Q^\times : \mathrm{tr}(\gamma\beta)=0\}\end{aligned}$$

being a difference set for $\mathcal{H}$ for all $\gamma\in\mathbb{F}_{Q^J}^\times$. As detailed in [8], this is indeed the case since, by properties of the field trace, this set is either $\mathcal{H}$ or a Singer difference set for $\mathcal{H}$.

Acknowledgments

The views expressed in this article are those of the authors and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the U.S. Government.

References

[1] J. H. Conway, R. H. Hardin, N. J. A. Sloane, Packing lines, planes, etc.: packings in Grassmannian spaces, Exp. Math. 5 (1996) 139–159.
[2] C. Ding, T. Feng, A generic construction of complex codebooks meeting the Welch bound, IEEE Trans. Inform. Theory 53 (2007) 4245–4250.
[3] I. S. Dhillon, J. R. Heath, T. Strohmer, J. A. Tropp, Constructing packings in Grassmannian manifolds via alternating projection, Exp. Math. 17 (2008) 9–35.
[4] M. Fickus, J. Jasper, E. J. King, D. G. Mixon, Equiangular tight frames that contain regular simplices, Linear Algebra Appl. 555 (2018) 98–138.
[5] M. Fickus, J. Jasper, D. G. Mixon, C. E. Watson, A brief introduction to equi-chordal and equi-isoclinic tight fusion frames, Proc. SPIE 10394 (2017) 103940T/1–9.
[6] M. Fickus, D. G. Mixon, Table of the existence of equiangular tight frames, arXiv:1504.00253 (2016).
[7] M. Fickus, D. G. Mixon, J. C. Tremain, Steiner equiangular tight frames, Linear Algebra Appl. 436 (2012) 1014–1027.
[8] M. Fickus, C. A. Schmitt, Harmonic equiangular tight frames comprised of regular simplices, in preparation.
[9] J. Jasper, D. G. Mixon, M. Fickus, Kirkman equiangular tight frames and codes, IEEE Trans. Inform. Theory. 60 (2014) 170-–181.
[10] D. Jungnickel, A. Pott, K. W. Smith, Difference sets, in: C. J. Colbourn, J. H. Dinitz (Eds.), CRC Handbook of Combinatorial Designs (2007) 419–435.
[11] T. Strohmer, R. W. Heath, Grassmannian frames with applications to coding and communication, Appl. Comput. Harmon. Anal. 14 (2003) 257–275.
[12] R. J. Turyn, Character sums and difference sets, Pacific J. Math. 15 (1965) 319–346.
[13] L. R. Welch, Lower bounds on the maximum cross correlation of signals, IEEE Trans. Inform. Theory 20 (1974) 397-–399.
[14] P. Xia, S. Zhou, G. B. Giannakis, Achieving the Welch bound with difference sets, IEEE Trans. Inform. Theory 51 (2005) 1900–1907.