A Delsarte-Style Proof of the Bukh–Cox Bound

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***Abstract*—The line packing problem is concerned with the optimal packing of points in real or complex projective space so that the minimum distance between points is maximized. Until recently, all bounds on optimal line packings were known to be derivable from Delsarte's linear program. Last year, Bukh and Cox introduced a new bound for the line packing problem using completely different techniques. In this paper, we use ideas from the Bukh–Cox proof to find a new proof of the Welch bound, and then we use ideas from Delsarte's linear program to find a new proof of the Bukh–Cox bound. Hopefully, these unifying principles will lead to further refinements.**

I. Introduction

The last decade has seen a surge of progress in *the line packing problem*, where the objective is to pack n points in $\mathbb{RP}^{d-1}$ or $\mathbb{CP}^{d-1}$ so that the minimum distance is maximized. Indeed, while instances of this problem date back to Tammes [1] and Fejes Tóth [2], the substantial progress in recent days has been motivated by emerging applications in compressed sensing [3], digital fingerprinting [4], quantum state tomography [5], and multiple description coding [6]. Most progress in this direction has come from identifying new packings that achieve equality in the so-called Welch bound (see [7] for a survey), but last year, Bukh and Cox [8] discovered a completely different bound, along with a large family of packings that achieve equality in their bound.

Focusing on the complex case, let $X = \{x_i\}_{i\in[n]}$ be a sequence of unit vectors in $\mathbb{C}^d$, and define coherence to be

$$\mu(X) := \max_{1\leq i<j\leq n} |\langle x_i, x_j\rangle|.$$

Then the Buhk–Cox bound guarantees

$$\mu(X) \geq \frac{(n-d)^2}{n+(n^2-nd-n)\sqrt{1+n-d}-(n-d)^2}$$

provided $n > d$. The Bukh–Cox bound is the best known lower bound on coherence in the regime where $n = d + O(\sqrt{d})$. While the other lower bounds can be proven using Delsarte's linear program [9], the proof of the Bukh–Cox bound is completely different: it hinges on an upper bound on the first moment of isotropic measures.

In the present paper, we provide an alternate proof of the Bukh–Cox bound. We start by isolating a lemma of Bukh and Cox that identifies a fundamental duality between the coherence of $n = d + k$ unit vectors in d dimensions and the entrywise 1-norm of the Gram matrix of $\frac{n}{k}$-tight frames of n vectors in k dimensions. Next, we illustrate the power of this lemma by using it to find a new proof of the Welch bound. Finally, we combine the lemma with ideas from Delsarte's linear program to obtain a new proof of the Bukh–Cox bound. This new proof helps to unify the existing theory of line packing, and hopefully, it will spur further improvements (say, by leveraging ideas from semidefinite programming [10]).

II. The Bukh–Cox Lemma

Let $X = \{x_i\}_{i=1}^n$ denote any sequence in $\mathbb{C}^d$. By abuse of notation, we identify X with the $d \times n$ matrix whose ith column is x_i. We say X is a *c-tight frame* if $XX^* = cI$. Let $N(d,n)$ denote the set of matrices in $\mathbb{C}^{d\times n}$ with unit norm columns, and let $T(d,n)$ denote the set of matrices in $\mathbb{C}^{d\times n}$ corresponding to $\frac{n}{d}$-tight frames. Define

$$\gamma(d,n) := \max_{Y\in T(d,n)} \|Y^*Y\|_1.$$

(Indeed, the maximum exists by a compactness argument.) We say $X \in N(d,n)$ is *equiangular* if there exists a constant c such that $|\langle x_i, x_j\rangle| = c$ for every $1 \leq i < j \leq n$. With this nomenclature, we are ready to state the following lemma of Bukh and Cox:

Lemma 1. *Let $n = d + k$. Then every $X \in N(d,n)$ satisfies*

$$\mu(X) \geq \frac{n}{\gamma(k,n)-n}. \tag{1}$$

Furthermore, X minimizes $\mu(X)$ over $N(d,n)$ if X is equiangular and there exists $Y = \{y_i\}_{i=1}^n \in T(k,n)$ such that

(i) *Y maximizes $\|Y^*Y\|_1$ over $T(k,n)$,*
(ii) *$XY^* = 0$, and*
(iii) *$\operatorname{sgn}\langle x_i, x_j\rangle = -\operatorname{sgn}\langle y_i, y_j\rangle$ for $1 \leq i < j \leq n$.*

Proof. Given $X \in N(d,n)$, select $Y \in T(k,n)$ such that $XY^* = 0$. Then,

$$\begin{aligned} 0 = (X^*XY^*Y)_{ii} &= \sum_{j=1}^n \langle x_i, x_j\rangle\langle y_j, y_i\rangle \\ &= \|y_i\|_2^2 + \sum_{\substack{j=1\\j\neq i}}^n \langle x_i, x_j\rangle\langle y_j, y_i\rangle. \end{aligned} \tag{2}$$

Bringing $\|y_i\|_2^2$ to the left hand side of (2) and taking absolute values, we have that

$$\|y_i\|_2^2 = \Bigg|\sum_{\substack{j=1\\j\neq i}}^n \langle x_i, x_j\rangle\langle y_j, y_i\rangle\Bigg| \leq \sum_{\substack{j=1\\j\neq i}}^n |\langle x_i, x_j\rangle||\langle y_i, y_j\rangle| \tag{3}$$

$$\leq \mu(X)\sum_{\substack{j=1\\j\neq i}}^n |\langle y_i, y_j\rangle|, \tag{4}$$

where (3) uses the triangle inequality and (4) is by the definition of coherence. Using (4) and $YY^* = (n/k)I$,

$$\begin{aligned} n &= \operatorname{tr}(YY^*) = \operatorname{tr}(Y^*Y) = \sum_{i=1}^{n} \|y_i\|_2^2 \\ &\leq \sum_{i=1}^{n} \mu(X) \sum_{\substack{j=1 \\ j\neq i}}^{n} |\langle y_i, y_j\rangle| = \mu(X)(\|Y^*Y\|_1 - \operatorname{tr}(Y^*Y)) \\ &= \mu(X)(\|Y^*Y\|_1 - n). \end{aligned}$$

Thus, we conclude that

$$\mu(X) \geq \frac{n}{\|Y^*Y\|_1 - n} \geq \frac{n}{\gamma(k,n) - n}. \tag{5}$$

This proves the bound. Considering (2), equality occurs in (3), (4) and (5) when X is equiangular and (i)–(iii) holds. □

III. The Welch Bound

Theorem 2. *For all $Y \in T(k,n)$ we have*

$$\|Y^*Y\|_1 \leq n + \left[n(n-1) \left(\frac{n^2}{k} - n \right) \right]^{1/2}. \tag{6}$$

Equality is achieved if and only if Y is an equiangular tight frame.

Proof. First we separate the diagonal part of $\|Y^*Y\|_1$ and use the Cauchy–Schwarz inequality,

$$\begin{aligned} \|Y^*Y\|_1 &= n + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} |\langle y_i, y_j\rangle| \\ &\leq n + \left[n(n-1) \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} |\langle y_i, y_j\rangle|^2 \right]^{1/2}. \end{aligned} \tag{7}$$

Noting that the the sum in (7) is (almost) $\|Y^*Y\|_F^2$ and once again using the Cauchy–Schwarz inequality,

$$\begin{aligned} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} |\langle y_i, y_j\rangle|^2 &= \|Y^*Y\|_F^2 - \sum_{i=1}^{n} \|y_i\|_2^4 \\ &= \operatorname{tr}(Y^*YY^*Y) - n \cdot \frac{1}{n} \sum_{i=1}^{n} (\|y_i\|_2^2)^2 \\ &\leq \frac{n^2}{k} - n \left(\frac{1}{n} \sum_{i=1}^{n} \|y_i\|_2^2 \right)^2 \qquad (8) \\ &= \frac{n^2}{k} - n. \qquad (9) \end{aligned}$$

Putting this back into (7) we obtain the inequality

$$\|Y^*Y\|_1 \leq n + \left[n(n-1) \left(\frac{n^2}{k} - n \right) \right]^{1/2}. \tag{10}$$

Equality is achieved in the Cauchy–Scwharz inequality if and only if the vectors are scalar multiples. For the first instance of Cauchy–Schwarz, this occurs if and only if Y is equiangular. For the second instance of Cauchy–Schwarz, equality is achieved if and only if $\|y_i\|_2^2$ is constant, that is $Y \in N(k,n)$. Thus, equality is achieved in (6) if and only if Y is an equiangular tight frame. □

Equality in (6) depends on the existence of equiangular tight frames of n vectors in $\mathbb{C}^k$. The Gerzon bound says that if Y forms an equiangular tight frame of n vectors in $\mathbb{C}^k$, then $n \leq k^2$ [7]. This gives the bound $k \geq 1/2 + \sqrt{1+4d}/2$ as a necessary condition for Y to be an equiangular tight frame. On the other hand, if Y in Lemma 1 is an equiangular tight frame, then X is also an equiangular tight frame since X and Y are Naimark complements [11]. In particular, this gives the upper bound $k \leq d^2 - d$ as a necessary condition for Y to be equiangular, because the Gerzon bound applied to X gives the requirement that $n \leq d^2$.

Being within the range $1/2 + \sqrt{1+4d}/2 \leq k \leq d^2 - d$ is not a sufficient condition. The existence of equiangular tight frames of n vectors in k dimensions for $k+1 \leq n \leq k^2$ is an open question. Some equiangular tight frames are known to exist, by construction, for certain values of n and k. An overview of the known constructions can be found in [7]. On the other hand, there are known values of n and k for which equiangular tight frames cannot exist. One such example is the case when $n = 8$ and $k = 3$ [12]. In particular, equality in the Welch bound is achieved for some values of n and k which satisfy $k+1 \leq n \leq k^2$, but not necessarily achieved at all values of n and k which satisfy that inequality.

By combining Lemma 1 with Theorem 2, we obtain

Corollary 3 (Welch)**.** *Let $n > d$. For all $X \in N(d,n)$,*

$$\mu(X) \geq \sqrt{\frac{n-d}{d(n-1)}}. \tag{11}$$

IV. Bukh–Cox Bound via Linear Programming

We now turn our attention to the range $1 \leq k < 1/2 + \sqrt{1+4d}/2$. Since the Gerzon bound prevents Y from being an equiangular tight frame in this range, equality in (6) cannot be achieved and a sharper bound can be obtained. The Bukh–Cox bound is an improvement in this range, and is sharp if Y is given by concatenated copies of k^2 vectors in $\mathbb{C}^k$ which forms an equiangular tight frame. In order to apply Delsarte's linear programming ideas, we require the following special polynomials [9]:

$$\begin{aligned} Q_0(x) &= 1, \\ Q_1(x) &= x - \frac{1}{k}, \\ Q_2(x) &= x^2 - \frac{4}{k+2}x + \frac{2}{(k+1)(k+2)}. \end{aligned}$$

Theorem 4. *For all $Y \in T(k,n)$ we have*

$$\|Y^*Y\|_1 \leq \frac{n^2(1 + (k-1)\sqrt{1+k})}{k^2}. \tag{12}$$

Equality is achieved when Y is of the form $Y = [Z|Z|\cdots|Z]$ where $Z \in \mathbb{C}^{k\times k^2}$ is an equiangular tight frame.

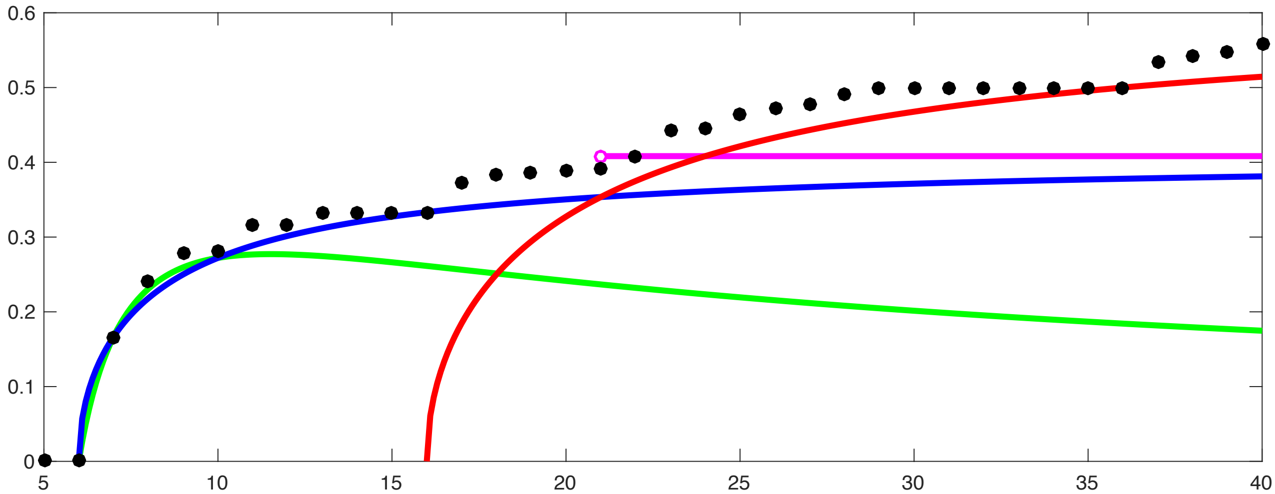

Fig. 1. The coherence of best known line packings in $\mathbb{R}^6$ for $n \in \{5, \ldots, 40\}$, along with the best known lower bounds. The black dots correspond to packings found in Sloane's database [13]. The Bukh–Cox bound is displayed in green, the Welch bound in blue, the orthoplex bound [14] in pink, and the Levenshtein bound [15] in red. In this setting, the Bukh–Cox bound is the best known lower bound for $n = \{8, 9\}$.

Proof. By continuity, we may assume $y_i \neq 0$ for every $i \in \{1, \cdots, n\}$ without loss of generality. First, we normalize the columns of Y, $\{y_i\}_{i=1}^n$, by defining $z_i := y_i/\|y_i\|_2$. We obtain the desired bound considering the following linear program, inspired by Delsarte's LP bound,

$$\begin{aligned} &\text{minimize } c_0 \\ &\text{subject to } f(x) = c_0 Q_0(x) + c_1 Q_1(x) + c_2 Q_2(x) \\ &\qquad\qquad 0 \le c_1 \le k c_0, \;\; c_2 \le 0, \\ &\qquad\qquad f(x) \ge \sqrt{x}, \;\; \forall x \in [0,1]. \end{aligned}$$

Suppose we have a feasible (c_0, c_1, c_2). Then,

$$\begin{aligned} \|Y^*Y\|_1 &= \sum_{i=1}^n \sum_{j=1}^n |\langle z_i, z_j\rangle| \, \|y_i\|_2 \, \|y_j\|_2 \\ &\le \sum_{i=1}^n \sum_{j=1}^n f(|\langle z_i, z_j\rangle|^2) \|y_i\|_2 \, \|y_j\|_2 \\ &= \sum_{\ell=0}^2 \sum_{i=1}^n \sum_{j=1}^n c_\ell Q_\ell(|\langle z_i, z_j\rangle|^2) \|y_i\|_2 \, \|y_j\|_2 \\ &= \sum_{\ell=0}^2 c_\ell \sum_{i=1}^n \sum_{j=1}^n Q_\ell(|\langle z_i, z_j\rangle|^2) \|y_i\|_2 \, \|y_j\|_2. \end{aligned} \tag{13}$$

We now establish an upper bound for each innermost term for $\ell \in \{0, 1, 2\}$. For $\ell = 0$, since $Q_0(x) = 1$, we have

$$\begin{aligned} &\sum_{i=1}^n \sum_{j=1}^n Q_0(|\langle z_i, z_j\rangle|^2) \|y_i\|_2 \, \|y_j\|_2 \\ &= \sum_{i=1}^n \sum_{j=1}^n \|y_i\|_2 \, \|y_j\|_2 = \left(\sum_{i=1}^n \|y_i\|_2 \right)^2 \\ &=: S. \end{aligned} \tag{14}$$

For $\ell = 1$, we have

$$\begin{aligned} &\sum_{i=1}^n \sum_{j=1}^n Q_1(|\langle z_i, z_j\rangle|^2) \|y_i\|_2 \, \|y_j\|_2 \\ &= \sum_{i=1}^n \sum_{j=1}^n \frac{|\langle y_i, y_j\rangle|^2}{\|y_i\|_2 \, \|y_j\|_2} - \frac{1}{k} \sum_{i=1}^n \sum_{j=1}^n \|y_i\|_2 \, \|y_i\|_2. \qquad (15) \\ &= \sum_{i=1}^n \sum_{j=1}^n \frac{|\langle y_i, y_j\rangle|^2}{\|y_i\|_2 \, \|y_j\|_2} - S/k. \qquad (16) \end{aligned}$$

To bound the first term of (16), we use Cauchy–Schwarz and the fact that Y is an n/k-tight frame:

$$\begin{aligned} &\sum_{i=1}^n \sum_{j=1}^n \frac{|\langle y_i, y_j\rangle|^2}{\|y_i\|_2 \, \|y_j\|_2} \\ &\le \left(\sum_{i=1}^n \sum_{j=1}^n \frac{|\langle y_i, y_j\rangle|^2}{\|y_i\|_2^2} \right)^{1/2} \left(\sum_{i=1}^n \sum_{j=1}^n \frac{|\langle y_i, y_j\rangle|^2}{\|y_j\|_2^2} \right)^{1/2} \\ &= \left(\sum_{i=1}^n \frac{n}{k} \frac{\|y_i\|_2^2}{\|y_i\|_2^2} \right)^{1/2} \left(\sum_{j=1}^n \frac{n}{k} \frac{\|y_j\|_2^2}{\|y_j\|_2^2} \right)^{1/2} = \frac{n^2}{k}. \end{aligned} \tag{17}$$

Overall this gives the following bound for the $\ell = 1$ case:

$$\sum_{i=1}^n \sum_{j=1}^n Q_1(|\langle z_i, z_j\rangle|^2) \|y_i\|_2 \, \|y_j\|_2 \le \frac{1}{k}(n^2 - S). \tag{18}$$

Last, we need a bound for the $\ell = 2$ case. Let $\{e_m\}_{m=1}^{d_2}$ be any orthonormal basis for the (finite) d_2-dimensional vector space spanned by degree-4 projective harmonic polynomials in k

variables. Then, by the addition formula, there is a constant $C_{d_2,k}$, which depends on d_2 and k, such that

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n}Q_2(|\langle z_i,z_j\rangle|^2)\|y_i\|_2\,\|y_j\|_2\\&=C_{d_2,k}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{d_2}e_m(z_i)\overline{e_m(z_j)}\|y_i\|_2\,\|y_j\|_2\\&=C_{d_2,k}\sum_{m=1}^{d_2}\left|\sum_{i=1}^{n}e_m(z_i)\|y_i\|_2\right|^2\geq 0.\end{aligned}\tag{19}$$

Multiplying both sides of (19) by $c_2 \leq 0$ then gives

$$\sum_{i=1}^{n}\sum_{j=1}^{n}c_2Q_2(|\langle z_i,z_j\rangle|^2)\|y_i\|_2\,\|y_j\|_2\leq 0.\tag{20}$$

Finally, returning to (13) we have

$$\begin{aligned}\|Y^*Y\|_1&=\sum_{\ell=0}^{2}c_\ell\sum_{i=1}^{n}\sum_{j=1}^{n}Q_\ell(|\langle z_i,z_j\rangle|^2)\|y_i\|_2\,\|y_j\|_2\\&\leq c_0S+c_1\frac{1}{k}(n^2-S)=\left(c_0-\frac{c_1}{k}\right)S+c_1\cdot\frac{n^2}{k}\\&\leq c_0n^2,\end{aligned}\tag{21}$$

where we have used that $S \leq n^2$. The bound (12) comes from observing that the following choice of (c_0, c_1, c_2) is feasible:

$$\begin{aligned}c_0&=\frac{1+(k-1)\sqrt{1+k}}{k^2}\\c_1&=\frac{\sqrt{1+k}(-4+k^2+4\sqrt{1+k})}{2k(2+k)}\\c_2&=\frac{-(2+4k+2k^2)+\sqrt{1+k}(2+3k+k^2)}{2k^2}.\end{aligned}$$

This feasible choice comes from forcing $f(\frac{1}{k+1}) = 1/\sqrt{k+1}$, $f(1) = 1$, and $f'(\frac{1}{k+1}) = \sqrt{k+1}/2$, and solving for (c_0, c_1, c_2). Equality is achieved in inequalities (13), (18), (20), (21) when

1) $|\langle y_i, z_j\rangle| = |\langle z_i, y_j\rangle|,\ \forall i, j$,
2) $f(|\langle z_i, z_j\rangle|^2) = |\langle z_i, z_j\rangle|,\ \forall i, j$,
3) $\sum_i \sum_j Q_2(|\langle z_i, z_j\rangle|^2)\|y_i\|_2\,\|y_j\|_2 = 0$,
4) $\|y_i\|_2 = 1,\ \forall i$.

All four are achieved if Y is multiple copies of an equiangular tight frame of k^2 vectors in $\mathbb{C}^k$. □

The proof of Theorem 4 actually generates a bound for any feasible (c_0, c_1, c_2) in the described linear program. Minimizing c_0 gives the best possible bound generated by this method. Computational experiments suggest that this particular feasible (c_0, c_1, c_2) gives the minimum c_0. Although equality is achieved when Y is multiple copies of an equiangular tight frame of k^2 vectors in $\mathbb{C}^k$, the existence of such frames is an open question, and is known as Zauner's conjecture [16].

By combining Lemma 1 with Theorem 4, we obtain

Corollary 5 (Bukh–Cox)**.** *Let $n > d$. For all $X \in N(d, n)$,*

$$\mu(X)\geq\frac{(n-d)^2}{n+(n^2-nd-n)\sqrt{1+n-d}-(n-d)^2}.$$

Bukh and Cox also provide a new bound for the case of n vectors in $\mathbb{R}^d$. For the real case, it suffices to adjust the special polynomials in the proof of Theorem 4 [9]. $Q_0(x)$ and $Q_1(x)$ remain the same, but $Q_2(x)$ is replaced with:

$$Q_2(x)=x^2-\frac{6}{d+4}x+\frac{3}{(d+2)(d+4)}.$$

This adjustment changes the feasible region for the linear program and leads to a different optimal (c_0, c_1, c_2), and thus a different bound for the $\mathbb{R}^d$ case. Fig. 1 demonstrates the Bukh–Cox bound improvement over the Welch bound for small values of k in the case where the vectors are in $\mathbb{R}^6$. (We illustrate the real case since, in this case, packing data is available and provided in [13].)

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