Scaling limits in planar eigenvalue ensembles

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***Abstract*—In this article I present some of my recent research on scaling limits in two-dimensional eigenvalue ensembles.**

I. Background

The infinite Ginibre ensemble provides the simplest example of a scaling limit of a two-dimensional random eigenvalue process. Another type of determinantal point field appears if we zoom at a boundary point as in Figure 1. The law of the limiting process is determined by Forrester-Honner's erfc-kernel.

In recent years, various types of determinantal point-fields appearing in similar ways, as scaling limits of normal eigenvalue ensembles have been identified and investigated, and a general theory is emerging. I will give a glimpse into these developments, as given for instance in [2], [3], [4], [5] and references.

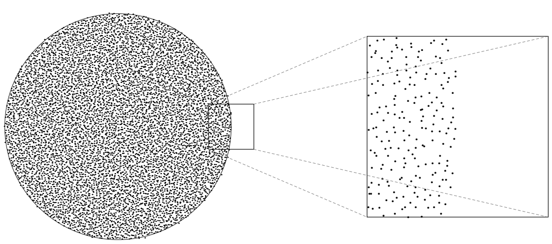

Fig. 1. A sample from the Ginibre ensemble, rescaled about a boundary point.

II. Basic setup

We begin with recalling the model. Fix a suitable external potential $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$, a number $\beta > 0$, and a large integer n. The corresponding Coulomb gas is an n-point configuration $\{\zeta_j\}_1^n$ picked randomly with respect to the Gibbs distribution

$$d\mathbf{P}_n^{(\beta)} = (Z_n^{(\beta)})^{-1}\, e^{-\beta H_n}\, dV_n,$$
$$H_n = \sum_{j \neq k} \log \frac{1}{|\,\zeta_j - \zeta_k\,|} + n \sum_{j=1}^{n} Q(\zeta_j).$$

The normalizing constant $Z_n^{(\beta)}$ is called the partition function; 'dV_n' is suitably normalized Lebesgue measure on $\mathbb{C}^n$. The Ginibre ensemble corresponds to $Q(\zeta) = |\zeta|^2$.

Given suitable conditions on Q, the system $\{\zeta_j\}_1^n$ tends to follow the *equilibrium measure* σ in external potential Q, in the sense that $\frac{1}{n}\mathbf{E}_n^{(\beta)}(f(\zeta_1)+\cdots+f(\zeta_n)) \to \sigma(f)$ as $n \to \infty$, for each continuous bounded function f. If Q is smooth in a neighbourhood of $\operatorname{supp}\sigma$, then σ is absolutely continuous and of the form $d\sigma = \Delta Q \cdot \mathbf{1}_S\, dA$ where $S := \operatorname{supp}\sigma$ is called the *droplet*. (Here $\Delta = \partial\bar{\partial}$ and 'dA' is the Lebesgue measure in the plane divided by π.)

The droplet depends on the potential in a nontrivial way, and can be characterized by means of the solution to an obstacle problem. It is well-known that the droplet corresponding to a smooth potential can be highly irregular. However, if the potential is *real-analytic* in a neighbourhood of the boundary ∂S, then this boundary is the union of finitely many real-analytic arcs, possibly with finitely many singular points of certain types, see Figure 2.

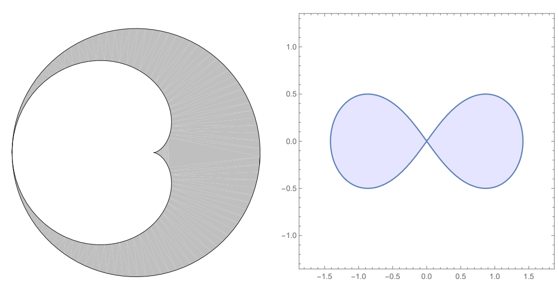

Fig. 2. Left: a droplet with one double point and one cusp on its boundary. Right: a boundary with a crossing point.

III. Determinantal processes

The special choice $\beta = 1$ plays a particular role. In this case, the process $\{\zeta_j\}_1^n$ is determinantal, making Bergman space techniques available. More precisely, a correlation kernel $\mathbf{K}_n$ can be constructed as the reproducing kernel for the space of weighted polynomials $w = qe^{-nQ/2}$ (degree $q \le n$) endowed with the topology of $L^2(\mathbb{C})$, by $\mathbf{K}_n(\zeta,\eta) = \sum w_j(\zeta)\bar{w}_j(\eta)$ where $\{w_j\}$ is an orthonormal basis.

Until recently, little was known in the way of general asymptotics of weighted orthogonal polynomials of this kind. However, there has now been progress in this direction due to the work of Hedenmalm and Wennman [8]. Furthermore, for certain important special ensembles, the technique of

Riemann-Hilbert problems has been applied, giving a very detailed information about orthogonal polynomials, and leading to new insights.

Now fix a (perhaps n-dependent) point $p \in S$ and rescale via $z_j = r_n^{-1}(\zeta - p)$, where r_n, the 'microscopic scale', is typically chosen to be proportional to $1/\sqrt{n\Delta Q(p)}$. (We use a coarser scale if $\Delta Q(p) = 0$.) Thus the rescaled system $\{z_j\}_1^n$ is obtained by placing the origin at p and blowing up distances by a factor $\text{const.}\sqrt{n}$.

The structure of weighted polynomials enables us to apply compactness arguments to prove existence and basic qualitative properties of limiting point fields $\{z_j\}_1^\infty$, see [3], [4], [5]. It is important to note that such a point field is uniquely determined by its one-point intensity function $R(z)$, which has the meaning of expected number of particles z_j per unit area at a point $z \in \mathbb{C}$.

A glance at Fig. 1 indicates that, for the Ginibre ensemble, we get very different results when we zoom on a bulk point or at a boundary point. The question of universality of the latter kernel, at regular boundary points of a droplet, was studied at length in [3]. The problem was later solved in a rather satisfactory generality in [8].The techniques in [3] and [8] are in a way complementary. In [5] they are combined to prove a kind of central limit theorem for $\log|p_n(\zeta)|$ where p_n is the characteristic polynomial of a random normal matrix with respect to an 'algebraic potential'.

IV. Ward's equation

A principal tool behind our approach is *Ward's identity* (or the *loop equation*). This is an exact relation connecting the 1- and 2-point functions of a β-ensemble. In the present context, it was used systematically by Wiegmann and Zabrodin and their school [6].

The one-dimensional counterpart to Ward's identity has been used, for example, in the influential paper [9] to study fluctuations in one-dimensional β-ensembles. It should however be noted that the Ward identities in dimension 1 and 2 are quite different from one other, analytically speaking.

Rescaling in Ward's identity, we obtain *Ward's equation*, which has the formal appearance

$$\bar{\partial} C = R - 1 - \frac{1}{\beta}\,\Delta \log R. \tag{1}$$

Here R is the 1-point function of a limiting point process and C is a certain related function, which is determined by the 1- and 2-point functions of a limiting ensemble.

Our program for studying this equation is easiest to describe in the case $\beta = 1$, because then C is determined by R in the following way. The correlation kernel K of the limiting determinantal process is determined by its diagonal values $K(z,z) = R(z)$, and provided that R does not vanish identically we have

$$C(z) = \int_{\mathbb{C}} \frac{1}{z-w}\frac{|K(z,w)|^2}{R(z)}\,dA(w).$$

As a consequence, Ward's equation (1) can be regarded as a 'closed' equation for the single unknown function R. Moreover, the kernel K can be shown to be the reproducing kernel of some contractively embedded subspace of a Fock-type space of weighted entire functions, encoding the local properties of the ensemble near the point p. In this way, we can interpret properties of our limiting point fields as statements about Hilbert spaces of entire functions. By contrast to the situation in Hermitian random matrix theory, the present spaces are not of de Branges type, but rather of Fock type.

In order to single out a particular solution R, we need additional information (e.g. decay properties) depending on the nature of the point we are zooming at. In the simplest case, when we zoom at a regular bulk point, it is not very hard to show that $R(0) = 1$. Using Ward's equation we can show that this implies $R \equiv 1$, which means that the limiting point field is just the infinite Ginibre ensemble.

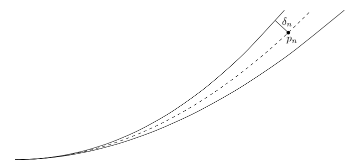

Fig. 3. Rescaling about the moving point p_n

V. Translation invariant solutions

Let us now assume that we are zooming on a moving point p_n approaching a cusp as in Figure 3. Here p_n is the point inside the droplet which is closest to the cusp, subject to the condition that the distance δ_n of p_n to the boundary is kept proportional to $1/\sqrt{n}$. Observe that the rescaled droplet in this case will look like a strip, whence it is very reasonable to look for solutions R which are *translation invariant*, i.e., invariant with respect to translations that preserve the strip. Given the assumption of translation invariance, Ward's equation can be reduced to a convolution equation, which in turn is completely solved using harmonic analysis in [3]. In fact, the general translation invariant solution is given by a convolution between a Gaussian and the characteristic function of an interval (or 'window'). See Fig. 4 for some density profiles (graphed w.r.t. a cross-section of the strip) obtained in this way, for various choices of width of the strip.

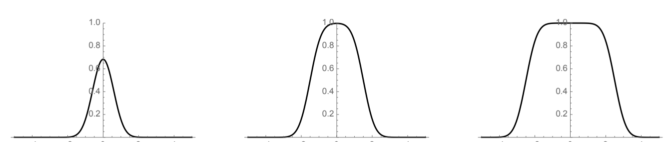

Fig. 4. Density profiles of translation invariant processes.

A parallel theory emerges by considering ensembles $\{\zeta_j\}_1^n$ with a *hard edge confinement*, i.e., when no particle is allowed to enter the complement of the droplet S. This is accomplished by redefining the potential Q to be $+\infty$ outside of S. Figure 5

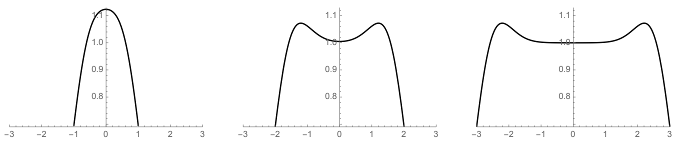

Fig. 5. Density profiles of translation invariant processes with hard edge confinement.

shows the density profiles of some hard edge point processes in strips of different widths.

It seems fair to say that the above indicated method solves the question of identifying limiting point fields near cusps on a 'physical' level. From a mathematical point of view, we must rigorously rule out the possibility of non translation invariant solutions to Ward's equation, which turns out to be a subtle matter. Without the assumption of translation invariance, Ward's equation becomes a (nonlinear) *twisted convolution equation* which is not yet fully understood, cf. [3, Section 8.3] and [7].

VI. Singularities in the bulk

Other situations occur when we rescale about singular points of different kinds, e.g. points in the bulk where the equilibrium density vanishes, and/or a conical singularity where the geometry degenerates. In these cases, Ward's equation takes on different forms, depending on the local behaviour of the Laplacian ΔQ near the singular point. When this behaviour is (asymptotically) rotationally symmetric, we obtain rotationally symmetric limiting point fields of the *Mittag-Leffler type*, see [5]. In the case of general bulk singularities, the density is believed to be expressible in terms of the Bergman kernel of a certain *Fock-Sobolev space* of entire functions.

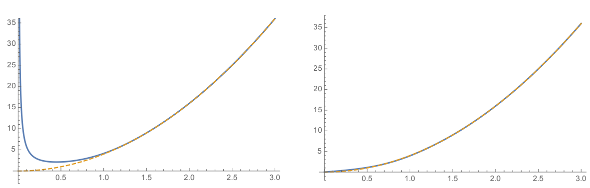

Fig. 6. Density profiles of some Mittag-Leffler type fields (in blue).

VII. Further examples of planar point fields

The *weakly Hermitian ensembles* are a family of processes that are concentrated near the real axis, and which interpolate in a natural way between 2-dimensional processes (such as the infinite Ginibre) and 1-dimensional ones (such as the sine-process). Cf. [1] and references.

The *lemniscate ensembles* are obtained by rescaling about a singularity of the lemniscate type, as in Fig. 2. Fig. 6 shows some numerically computed level curves of the rescaled one-point density $R(z)$ about the singular point. (Explicitly, the potential is $Q(\zeta) = |\zeta|^4 - \sqrt{2}\,\mathrm{Re}(\zeta^2)$ and the droplet is the domain enclosed by the lemniscate $|\zeta^2 - 1/\sqrt{2}| = 1/\sqrt{2}$ which has a two-fold crossing point at the origin. Cf. [4] for more in this connection.)

Another kind of point field is obtained by inserting a point charge of strength c at a boundary point of the droplet and studying the conditional process, for which eigenvalues are repelled in a nontrivial way from this point, see Fig. 8. In this case, a Dirac point mass of strength c enters Ward's equation, which must now be interpreted in a distributional sense.

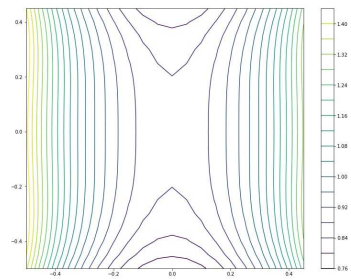

Fig. 7. Level curves of the density of a lemniscate field of order 2. The picture uses an approximation by the 1-point function corresponding to $n = 600$ particles.

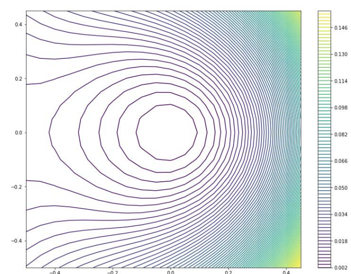

Fig. 8. Local density of a Ginibre process conditioned on insertion of a repelling charge at a boundary point.

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