The Diamond ensemble: a well distributed family of points on $\mathbb{S}^2$

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***Abstract*—We present our recent work on the Diamond ensemble, a constructive family of points on $\mathbb{S}^2$ defined in [1]. Among the various characteristic of the Diamond ensemble we stand out that every set of points of the family is described by very simple formulas and that we can compute analytically also the expectation of the logarithmic energy of those sets of points. The value that we obtain for the logarithmic energy is by far, among the ones that have been proved for constructible sequences, the minimal to the date (see [2]).**

***Index Terms*—spherical points, minimal logarithmic energy**

I. Introduction

Well distributed points on the sphere $\mathbb{S}^2$ conform an interesting object of study that has attracted researchers from very different fields of mathematics. See [8] and [9] for some of the principal questions concerning these families of points. In the survey [7] the authors present a variety of families of points on $\mathbb{S}^2$ together with an extensive experimental study of some of their properties.

However, for all those families it has proved quite difficult to obtain theoretical values for classical measures of the quality of discrete distributions. In particular, there are no known results on these families for our main object of study –the logarithmic energy.

The *Diamond ensemble* defined in [1] is a collection of random points depending on several parameters. For appropriate choices of the parameters, this construction produces families of points that very much resemble some already known families for which the asymptotic expansion of the logarithmic energy is unknown, such as the Octahedral points or the Zonal Equal Area Nodes, see [5], [6] and [7].

II. Well distributed points

There exist several characterizations for well distributed points on $\mathbb{S}^2$. For example, one can try to see how much the associated discrete measure resembles the uniform distribution in the set. We say that a set of N points $\omega_N = \{x_1, ..., x_N\}$ in $\mathbb{S}^2$, is asymptotically uniformly distributed if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} f(x_j) = \frac{1}{4\pi}\int_{\mathbb{S}^2} f(p)dp$$

for every continuous function $f : \mathbb{S}^2 \longrightarrow \mathbb{R}$. This definition is equivalent to the statement

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \chi(x_j) = \mu(C)$$

where χ is the indicator function of C for every continuous subset $C \subset \mathbb{S}^2$, where μ is the Lebesgue measure. The definition of asymptotically uniformly distributed can be analyzed on a much more general space, say any compact Hausdorff space X together with a non negative regular Borel measure, see [11, Chapter 3].

Another very popular way to describe how well distributed a finite set of spherical points is, is given by trying to minimize some energy depending on the points. It is of particular interest to study points that minimize the Riesz s-potential, i.e. $\omega_N = \{x_1, ..., x_N\} \subset \mathbb{S}^2$ such that ω_N minimizes

$$\mathcal{E}_s(\omega_N) = \sum_{i\neq j} \frac{1}{||x_i - x_j||^s}$$

among all the possible configurations of N points of $\mathbb{S}^2$. Such collections of points have been proved to be asymptotically uniformly distributed. See [12] and [13] for the cases $0 \le s \le 2$, and [14] for $s > 2$.

III. Logarithmic energy

If we let $s \to 0$ and take the derivative of $\mathcal{E}_s(\omega_N)$ we get the so–called logarithmic energy:

$$\mathcal{E}_{\log}(\omega_N) = \sum_{i\neq j} \log \frac{1}{||x_i - x_j||}.$$

Points that minimize the logarithmic energy satisfy many nice properties. As minimizers of Riesz energy, they are also asymptotically uniformly distributed, as proved in [16], and they are related to the condition number of a complex polynomial on one variable, see [17]. The problem of describing the asymptotic for the minimal possible value of the logarithmic energy of N points, m_N, is a fundamental open question in Potential Theory. The last word until moment has been given in [10] where this value is related to the minimum renormalized energy introduced in [15] proving the existence of an $O(N)$ term. The current knowledge is:

$$m_N = W_{\log}(\mathbb{S}^2)\, N^2 - \frac{1}{2}\, N \log N + C_{\log}\, N + o(N), \quad (1)$$

where

$$W_{\log}(\mathbb{S}^2)=\frac{1}{(4\pi)^2}\int_{x,y\in\mathbb{S}^2}\log\|x-y\|^{-1}\,d(x,y)=\frac{1}{2}-\log 2$$

is the continuous energy and $C_{\log}$ is a constant. Combining [16] with [10] it is known that

$$\begin{aligned}&-0.2232823526\ldots\leq C_{\log},\\ &C_{\log}\leq 2\log 2+\frac{1}{2}\log\frac{2}{3}+3\log\frac{\sqrt{\pi}}{\Gamma(1/3)}=-0.0556053\ldots,\end{aligned} \tag{2}$$

and indeed the upper bound for $C_{\log}$ has been conjectured to be an equality using two different approaches [4], [10].

IV. The Diamond ensemble

Fix $z\in(-1,1)$. The parallel of height z in the sphere $\mathbb{S}^2\subset\mathbb{R}^3$ is simply the set of points $x\in\mathbb{S}^2$ such that $\langle x,(0,0,1)\rangle=z$. A general construction of points can then be done as follows:

1) Choose a positive integer p and $z_1,\ldots,z_p\in(-1,1)$. Consider the p parallels with heights $z_1,\ldots,z_p$.
2) For each j, $1\leq j\leq p$, choose a number r_j of points to be allocated on parallel j.
3) Allocate r_j points in parallel j (which is a circumference) by projecting the r_j roots of unity onto the circumference and rotating them by random phase $\theta_j\in[0,2\pi]$.
4) To the already constructed collection of points, add the North and South pole $(0,0,1)$ and $(0,0,-1)$.

We will denote this random set by $\Omega(p,r_j,z_j)$. Explicit formulas for this construction are easily produced: points in parallel of height z_j are of the form

$$x_j=\left(\sqrt{1-z_j^2}\cos\theta,\sqrt{1-z_j^2}\sin\theta,z_j\right)$$

for some $\theta\in[0,2\pi]$ and thus the set we have described agrees with the following definition.

Let $\Omega(p,r_j,z_j)=\{\mathbf{n},\mathbf{s}\}\cup\{x_j^i\}$ with $\mathbf{n}=(0,0,1)$, $\mathbf{s}=(0,0,-1)$ and

$$x_j^i=\left(\sqrt{1-z_j^2}\cos\left(\frac{2\pi i}{r_j}+\theta_j\right),\sqrt{1-z_j^2}\sin\left(\frac{2\pi i}{r_j}+\theta_j\right),z_j\right) \tag{3}$$

where r_j is the number of roots of unity that we consider in the parallel j, $1\leq j\leq p$ is the number of parallels, $1\leq i\leq r_j$ and $0\leq\theta_j<2\pi$ is a random angle rotation in the parallel j.

The following result, [1, Proposition 2.5] has a simple proof.

Proposition 4.1: Given $\{r_1,...,r_p\}$ such that $r_i\in\mathbb{N}$, there exists a unique set of heights $\{z_1,\ldots,z_p\}$ such that $z_1>\ldots>z_p$ and $\mathrm{E}_{\theta_1,\ldots,\theta_p\in[0,2\pi]^p}\left[\mathcal{E}_{\log}(\Omega(p,r_j,z_j))\right]$ is minimized. The heights are:

$$z_l=\frac{\displaystyle\sum_{j=l+1}^{p}r_j-\sum_{j=1}^{l-1}r_j}{\displaystyle 1+\sum_{j=1}^{p}r_j}=1-\frac{1+r_l+2\sum_{j=1}^{l-1}r_j}{N-1},$$

where $N=2+\sum_{j=1}^{p}r_j$ is the total number of points.

Let p,M be two positive integers with $p=2M-1$ odd and let $r_j=r(j)$ where $r:[0,2M]\to\mathbb{R}$ is a continuous piecewise linear function satisfying $r(x)=r(2M-x)$ and

$$r(x)=\begin{cases}\alpha_1+\beta_1x & \text{if } 0=t_0\leq x\leq t_1\\ \vdots & \vdots\\ \alpha_n+\beta_nx & \text{if } t_{n-1}\leq x\leq t_n=M\end{cases}.$$

Here, $[t_0,t_1,\ldots,t_n]$ is some partition of $[0,M]$ and all the $t_\ell,\alpha_\ell,\beta_\ell$ are assumed to be integer numbers.

We assume that $\alpha_1=0$, $\alpha_\ell,\beta_\ell\geq 0$ and $\beta_1>0$ and there exists a constant $A\geq 2$ not depending on M such that $\alpha_\ell\leq AM$ and $\beta_\ell\leq A$. We also assume that $t_1\geq cM$ for some $c\geq 0$. Moreover, let z_j be as defined in Proposition 4.1.

We call the set of points defined this way the *Diamond ensemble* and we denote it by $\diamond(N)$, omiting in the notation the dependence on all the parameters $n,t_1,\ldots,t_n,\alpha_1,\ldots,\alpha_n$, $\beta_1,\ldots,\beta_n$. Note that the total number of points is

$$N=2-(\alpha_n+\beta_nM)+2\sum_{\ell=1}^{n}\sum_{j=t_{\ell-1}+1}^{t_\ell}(\alpha_\ell+\beta_\ell j).$$

We also denote by N_ℓ the total number of points in up to $t_{\ell-1}$, that is

$$N_\ell=\sum_{j=1}^{t_{\ell-1}-1}r_j.$$

Note that if $j\in[t_{\ell-1},t_\ell]$ then

$$\begin{aligned}z_j&=1-\frac{1+r_j+2\sum_{k=1}^{j-1}r_k}{N-1}\\ &=1-\frac{1+2N_j-r_j+2\sum_{k=t_{\ell-1}}^{j}(\alpha_\ell+\beta_\ell k)}{N-1}\\ &=1-\frac{1+2N_j-(\alpha_\ell+\beta_\ell j)+2\alpha_\ell(j-t_{\ell-1}+1)}{N-1}\\ &\qquad-\frac{\beta_\ell(j+t_{\ell-1})(j-t_{\ell-1}+1)}{N-1}.\end{aligned}$$

We thus consider the function $z(x)$ piecewise defined by the degree 2 polynomial

$$\begin{aligned}z_\ell(x)=1-\frac{1+2N_j-(\alpha_\ell+\beta_\ell x)+2\alpha_\ell(x-t_{\ell-1}+1)}{N-1}\\ -\frac{\beta_\ell(x+t_{\ell-1})(x-t_{\ell-1}+1)}{N-1}.\end{aligned}$$

V. Resemblance to other families of spherical points

In [5] Rakhmanov et al. define a diameter bounded, equal area partition of $\mathbb{S}^2$ consisting on two spherical caps on the south and the north pole and rectiliniar cells located on rings of parallels. The resemblance between our model and this model is remarkable, and even if the constructions are different, the points obtained seem to be really close. Both in [5] and in our work, a good deal of the effort is devoted to find choices of the values of r_j that produce sensible output and are integer numbers.

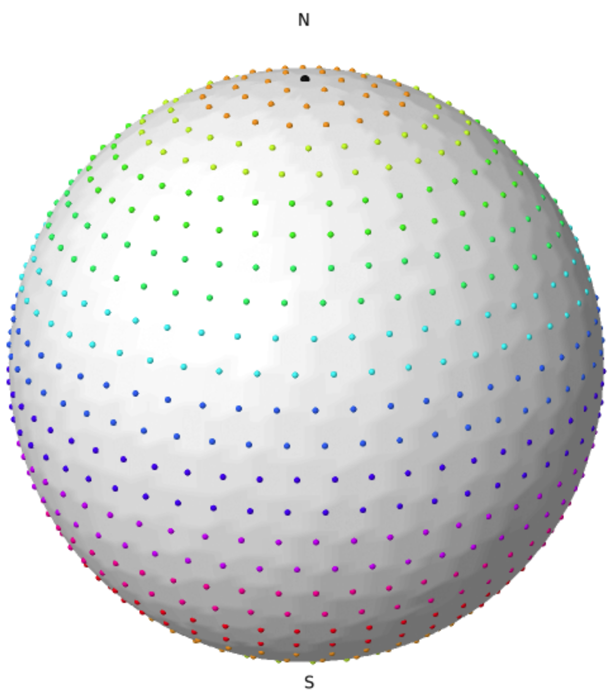

Fig. 1. A realization of the quasioptimal Diamond ensemble. Different colors correspond to the different domains of the linear functions defining $r(x)$.

Our main result (Theorem 6.1 below) is a theoretical bound for the logarithmic energy of points coming from the Diamond ensemble. Indeed, our bounds are slightly better than the numerical bounds obtained in [7] for the zonal equal area nodes.

In [6] an area preserving map from the unit sphere to the regular octahedron is defined. Considering some hierarchical triangular grids on the facets of the octahedron a grid can be mapped into the sphere obtaining two different sets of points: (using the notation from [6]) those coming from the vertex of the grid, Ω_N, and the centers of the triangles, Λ_N. Ω_N consists on $4M^2+2$ points in the sphere that can be seen as a concrete example (with fixed angles) of our general Diamond ensemble. In the paper, the authors give some numerical simulations for the logarithmic energy of this set of points that are confirmed by our results. Also in [7, Figure 2.2] new numerical simulations for the same set are done obtaining a bound which is very similar to the one we prove here.

VI. Logarithmic energy of the Diamond ensemble

For any given choice of the piecewise linear mapping defining $r(x)$ (and as far as the aforementioned hypotheses on the parameters are satisfied) one can obtain the value for the expected value of the logarithmic energy. We have performed an intensive search for piecewise approximations to the curve

$$\tilde{r}(x) = \frac{3\sin(x\pi/(p+1))}{\sin(\pi/(2p+1)},$$

which seems to be, by a heuristic argument, a good choice for an ideal $r(x)$, the drawback being that of course $\tilde{r}(j)$ is not integer for integer j. A quasioptimal choice of these parameters that approximates that curve quite closely and at the same time satisfies $r(j) \in \mathbb{Z}$ for $j \in \mathbb{Z}$ is now described. Let

$$r(x) = \begin{cases} 6x & 0 \le x \le 2m \\ 2m+5x & 2m \le x \le 3m \\ 5m+4x & 3m \le x \le 4m \\ 9m+3x & 4m \le x \le 5m \\ 14m+2x & 5m \le x \le 6m \\ 20m+x & 6m \le x \le 7m \\ 34m-x & 7m \le x \le 8m \\ 42m-2x & 8m \le x \le 9m \\ 51m-3x & 9m \le x \le 10m \\ 61m-4x & 10m \le x \le 11m \\ 72m-5x & 11m \le x \le 12m \\ 84m-6x & 12m \le x \le 14m = p+1 \end{cases}$$

and let z_l be the associated points given by Proposition 4.1. We call the resulting set the *quasioptimal Diamond ensemble*, and its main interest is that we can prove the following bound.

Theorem 6.1: The expected value of the logarithmic energy of the quasioptimal Diamond ensemble is

$$W_{\log}(\mathbb{S}^2)\, N^2 - \frac{1}{2}\, N \log N + c_\diamond\, N + o(N),$$

where $c_\diamond = -0.0492220914515784\ldots$ satisfies

$$\begin{aligned} 14340\, c_\diamond = {} & 19120 \log 239 - 2270 \log 227 - 1460 \log 73 \\ & - 265 \log 53 - 1935 \log 43 - 930 \log 31 - 1710 \log 19 \\ & - 1938 \log 17 + 19825 \log 13 + 1750 \log 7 - 4250 \log 5 \\ & - 131307 \log 3 + 56586 \log 2 - 7170. \end{aligned}$$

The value of the constant is thus approximately 0.0058 far from the value conjectured in (2). The Diamond ensemble is fully constructive: once a set of parameters is chosen, one just has to choose some uniform random numbers $\theta_1, \ldots, \theta_p \in [0, 2\pi]$ and then the N points are simply given by the direct formula (3) shown in Section IV. It is thus extremely easy to generate these sequences of points.

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