Average sampling and average splines on combinatorial graphs

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***Abstract*—In the setting of a weighted combinatorial finite or infinite countable graph G we introduce functional Paley-Wiener spaces $PW_{\omega}(L)$, $\omega > 0$, defined in terms of the spectral resolution of the combinatorial Laplace operator L in the space $L_2(G)$. It is shown that functions in certain $PW_{\omega}(L)$, $\omega > 0$, are uniquely defined by their averages over some families of "small" subgraphs which form a cover of G. Reconstruction methods for reconstruction of an $f \in PW_{\omega}(L)$ from appropriate set of its averages are introduced. One method is using language of Hilbert frames. Another one is using average variational interpolating splines which are constructed in the setting of combinatorial graphs.**

I. Introduction and main results

During the last decade signal processing on graphs was developed in a number of papers, for example, in [1], [4], [6], [8], [12]- [20]. Many of the papers on this list considered what can be called as a "point-wise sampling". The goal of the present article is to develop sampling on graphs which is based on averages over relatively small subgraphs. The idea to use local information for reconstruction of bandlimited functions on graphs was already explored in [19] and [4]. However, the results and methods of these papers and of our paper are very different. We also want to mention that methods of the present paper are similar to methods of our paper [11] in which sampling by average values was developed on Riemannian manifolds.

Let G denote an undirected weighted graph, with a finite or countable number of vertices $V(G)$ and weight function $w : V(G) \times V(G) \mapsto \mathbf{R}_0^+$. w is symmetric, i.e., $w(u, v) = w(v, u)$, and $w(u, u) = 0$ for all $u, v \in V(G)$. The edges of the graph are the pairs (u, v) with $w(u, v) \neq 0$.

Our assumption is that for every $v \in V(G)$ the following finiteness condition holds

$$w(v) = \sum_{u \in V(G)} w(u, v) < \infty. \tag{1}$$

Let $\ell^2(G)$ denote the space of all complex-valued functions with the inner product

$$\langle f, g \rangle = \sum_{v \in V(G)} f(v)\overline{g(v)}$$

and the norm

$$\|f\| = \left(\sum_{v \in V(G)} |f(v)|^2 \right)^{1/2}.$$

Definition 1: The weighted gradient norm of a function f on $V(G)$ is defined by

$$\|\nabla f\| = \left(\sum_{u,v \in V(G)} \frac{1}{2} |f(u) - f(v)|^2 w(u, v) \right)^{1/2}.$$

The set of all $f : G \mapsto \mathbf{C}$ for which the weighted gradient norm is finite will be denoted as $\mathcal{D}(\nabla)$.

Remark 1.1: The factor $\frac{1}{2}$ makes up for the fact that every edge (i.e., every *unordered* pair (u, v)) enters twice in the summation. Note also that loops, i.e. edges of the type (u, u), in fact do not contribute.

II. The gradient, the Laplace operator and the Paley-Wiener spaces.

In the case of a *finite* graph and $\ell^2(G)$-space the weighted Laplace operator $L : \ell^2(G) \to \ell^2(G)$ is introduced via

$$(Lf)(v) = \sum_{u \in V(G)} (f(v) - f(u))w(v, u) . \tag{2}$$

This graph Laplacian is a well-studied object; it is known to be a positive-semidefinite self-adjoint *bounded* operator. It is known (see [3], [5]) that if for an *infinite* graph there exists a $C > 0$ such that the degrees are uniformly bounded

$$w(u) = \sum_{u \in V(G)} w(u, v) \leq C, \tag{3}$$

then operator which is defined by (2) on functions with compact supports has a unique positive-semidefinite self-adjoint *bounded* extension L which is acting according to (2).

We use the spectral theorem for this operator L to introduce the associated Paley-Wiener spaces [9], [10].

Definition 2: $PW_{\omega}(L) \subset \ell^2(G)$ denote the image space of the projection operator $\mathbf{1}_{[0,\,\omega]}(L)$ (to be understood in the sense of Borel functional calculus).

By using the Spectral theorem one can show [9] that a function f belongs to the space $PW_{\omega}(L)$ if and only if for every positive $t > 0$ the following Bernstein-type inequality holds

$$\|L^t f\|_2 \leq \omega^t \|f\|_2, \quad t > 0. \tag{4}$$

Lemma 2.1: ([1], [7]) For all $f \in \ell^2(G)$ we have

$$\|L^{1/2} f\|^2 = \|\nabla f\|^2 . \tag{5}$$

For $f \in PW_\omega(L)$, this implies

$$\|\nabla f\| = \|L^{1/2} f\| \leq \sqrt{\omega}\|f\|. \tag{6}$$

Proof: We obtain

$$\begin{aligned} \langle f, Lf \rangle &= \sum_{u \in V(G)} f(u) \overline{\left(\sum_{v \in V(G)} (f(u) - f(v)) \, w(u,v) \right)} \\ &= \sum_{u \in V(G)} \left(|f(u)|^2 w(u) - \sum_{v \in V(G)} f(u) \overline{f(v)} w(u,v) \right) \end{aligned}$$

In the same way one obtains that $\langle Lf, f \rangle$ equals

$$\sum_{u \in V(G)} \left(|f(u)|^2 w(u) - \sum_{v \in V(G)} \overline{f(u)} f(v) w(u,v) \right) .$$

Averaging these equations yields

$$\langle f, Lf \rangle = \sum_{u,v \in V(G)} \frac{1}{2} |f(v) - f(u)|^2 w(u,v) = \|\nabla f\|^2 .$$

Now the first equality follows by taking the square root of L (note that by spectral theory, f is also in the domain of $L^{1/2}$), and (6) is an obvious consequence. ∎

III. A GLOBAL POINCARE-TYPE INEQUALITY FOR FINITE GRAPHS

It is well known that for every finite connected graph has $\lambda_0 = 0$ as a simple eigenvalue of the Laplace operator L and the corresponding eigenfunction is a constant on the entire graph. Given a connected and finite graph G and a function $f \in \ell^2(G)$ we consider its average

$$f_G = \frac{1}{|G|} \sum_{v \in V(G)} f(v),$$

where $|G|$ is the total number of vertices in $V(G)$. The notation $a\mathbf{1}$ is used for a constant function $f(v) = a$ for all $v \in G$.

Theorem 3.1: For every connected and finite graph G (which contains more than one vertex) the following Poincare inequality holds for $f \in \ell^2(G)$

$$\sum_{u \in V(G)} |f(u) - f_G \mathbf{1}|^2 \leq \frac{1}{\lambda_1} \|\nabla f\|^2 = \frac{1}{\lambda_1} \|L^{1/2} f\|^2, \tag{7}$$

where λ_1 is the first non-zero eigenvalue of L.

Proof: Note, that the average of the function $f - f_G \mathbf{1}$ is zero:

$$\sum_{u \in V(G)} \left(f(u) - \left(\frac{1}{|G|} \sum_{v \in V(G)} f(v) \right) \mathbf{1} \right) =$$
$$\sum_{u \in V(G)} f(u) - \sum_{v \in V(G)} f(v) = 0.$$

If λ_1 is the first nonzero eigenvalue of L then $\sqrt{\lambda_1}$ is the first nonzero eigenvalue of the nonnegative square root $L^{1/2}$. Since function $f - f_G$ is orthogonal to constants it implies

$$\|f - f_G \mathbf{1}\| \leq \frac{1}{\sqrt{\lambda_1}} \|L^{1/2}(f - f_G \mathbf{1})\| = \frac{1}{\sqrt{\lambda_1}} \|L^{1/2} f\|. \tag{8}$$

But according to Lemma 2.1 it gives

$$\|f - f_G \mathbf{1}\| \leq \frac{1}{\sqrt{\lambda_1}} \|\nabla f\|.$$

Theorem is proven. ∎

IV. A GENERALIZED POINCARE-TYPE INEQUALITY FOR FINITE AND INFINITE GRAPHS

Let G be a finite or infinite and countable connected graph and $\Omega \subset V(G)$ is a finite and connected subset of vertices which we will treat as an **induced** graph and will denote by the same letter Ω. We remind that this means that the set of vertices of such graph, which will be denoted as $V(\Omega)$, is exactly the set of vertices in Ω and the set of edges are all edges in G whose both ends belong to Ω. Let Δ_Ω be the Laplace operator constructed according to (2) for such induced graph Ω. The first nonzero eigenvalue of the operator operator Δ_Ω will be denoted as $\lambda_{1,\Omega}$. Let $w_\Omega(u,v), \; u,v \in V(\Omega)$, and

$$w_\Omega(v) = \sum_{u \in V(\Omega)} w_\Omega(u,v), \;\; v \in V(\Omega),$$

be the corresponding weight functions. We notice that for every Ω and every $u, v \in V(\Omega)$ one has $w(u,v) = w_\Omega(u,v)$. However, in general $w(u) \geq w_\Omega(u)$.

Suppose that $\Xi = \{\Omega_j\}$ is a disjoint cover of $V(G)$ by connected and finite subgraphs Ω_j. We define functions ξ_j by the formula

$$\xi_j = \frac{1}{\sqrt{|\Omega_j|}} \chi_j,$$

where χ_j is the characteristic function of Ω_j, and $|\Omega_j|$ is the number of vertices in Ω_j. We will be interestead in functionals on $\ell^2(G)$ defined by these functions

$$f \mapsto \langle f, \xi_j \rangle = \frac{1}{\sqrt{|\Omega_j|}} \sum_{v \in V(\Omega_j)} f(v), \quad f \in \ell^2(G).$$

We will also need functions

$$\zeta_j = \frac{1}{|\Omega_j|} \chi_j,$$

and corresponding functionals

$$f \mapsto \langle f, \zeta_j \rangle = \frac{1}{|\Omega_j|} \sum_{v \in V(\Omega_j)} f(v), \quad f \in \ell^2(G).$$

By using these notations we formulate the next theorem.

Theorem 4.1: Let G be a connected finite or infinite and countable graph. Suppose that $\Xi = \{\Omega_j\}$ is a disjoint cover of $V(G)$ by connected and finite subgraphs Ω_j. Let Δ_j be the Laplace operator of the **induced** graph Ω_j whose first nonzero eigenvalue is $\lambda_{1,j}$. We assume that that there exists a non zero lower boundary of all $\lambda_{1,j}$:

$$\Lambda = \Lambda(\Xi) = \inf_j \lambda_{1,j} > 0.$$

In these notations the following inequality holds for every $f \in \ell^2(G)$ and every $\alpha > 0$

$$\|f\|^2 \le \frac{1+\alpha}{\alpha}\frac{1}{\Lambda(\Xi)}\|L^{1/2}f\|^2 + (1+\alpha)\sum_j |\langle f, \xi_j\rangle|^2 . \quad (9)$$

Proof: One has

$$\|f\|^2 = \sum_{v\in V(G)} |f(v)|^2 = \sum_j \left(\sum_{v\in V(\Omega_j)} |f(v)|^2 \right). \quad (10)$$

For every $u \in V(\Omega_j)$ we apply the next inequality in which $f_{\Omega_j} = \langle f, \zeta_j\rangle$

$$|f(u)|^2 \le$$

$$\frac{1+\alpha}{\alpha}\left|f(u) - f_{\Omega_j}\chi_j(u)\right|^2 + (1+\alpha)\left|f_{\Omega_j}\chi_j(u)\right|^2, \quad (11)$$

which holds true for every positive $\alpha > 0$. According to Theorem 3.1

$$\sum_{u\in V(\Omega_j)} |f(u) - f_{\Omega_j}\chi_j(u)|^2 \le \frac{1}{\lambda_{1,j}}\|\nabla_j f\|^2, \quad (12)$$

where

$$\|\nabla_j f\|^2 = \sum_{u,v\in V(\Omega_j)} \frac{1}{2}\left|f(u) - f(v)\right|^2 w_j(u,v).$$

Thus we obtain

$$\sum_{u\in V(\Omega_j)} |f(u)|^2 \le \frac{1+\alpha}{\alpha}\sum_{u\in V(\Omega_j)} |f(u) - f_{\Omega_j}\chi_j(u)|^2 +$$

$$(1+\alpha)\sum_{u\in V(\Omega_j)} \left|f_{\Omega_j}\chi_j(u)\right|^2 \le$$

$$\frac{1+\alpha}{\alpha}\frac{1}{\lambda_{1,j}}\|\nabla_j f\|^2 + (1+\alpha)|\Omega_j|\left|f_{\Omega_j}\right|^2. \quad (13)$$

We can rewrite (13) in the following form

$$\|f\|^2 \le \frac{1+\alpha}{\alpha}\frac{1}{\Lambda(\Xi)}\sum_j \|\nabla_j f\|^2 + (1+\alpha)\sum_j |\Omega_j|\,|\langle f, \zeta_j\rangle|^2 =$$

$$\frac{1+\alpha}{\alpha}\frac{1}{\Lambda(\Xi)}\sum_j \left(\sum_{u,v\in V(\Omega_j)} \frac{1}{2}|f(u) - f(v)|^2 w_j(u,v) \right) +$$

$$(1+\alpha)\sum_j |\langle f, \xi_j\rangle|^2, \quad \alpha > 0. \quad (14)$$

Since for all j one has that $w(u,v) = w_j(u,v)$, $u, v \in V(\Omega_j)$, it is obvious that the first term in the last line is not greater than

$$\frac{1+\alpha}{\alpha}\frac{1}{\Lambda(\Xi)}\|\nabla f\|^2.$$

It gives for $\alpha > 0$

$$\|f\|^2 \le \frac{1+\alpha}{\alpha}\frac{1}{\Lambda(\Xi)}\|\nabla f\|^2 + (1+\alpha)\sum_j |\langle f, \xi_j\rangle|^2 ,$$

and by applying Theorem 2 we obtain (9). Theorem is proved. ■

V. A SAMPLING THEOREM AND A RECONSTRUCTION METHODS USING FRAMES

Theorem 5.1: If the assumptions of the previous Theorem hold then the set of functionals $\{\xi_j\}$ is a frame in any space $PW_\omega(L)$ as long as $\Lambda(\Xi) > \frac{1+\alpha}{\alpha}\omega$. To be more specific, if

$$\gamma = \frac{1+\alpha}{\alpha}\frac{\omega}{\Lambda(\Xi)} < 1, \quad \alpha > 0, \quad (15)$$

then

$$\frac{(1-\gamma)}{(1+\alpha)}\|f\|^2 \le \sum_j |\langle f, \xi_j\rangle|^2 \le \|f\|^2. \quad (16)$$

Proof: Indeed, if $f \in PW_\omega(L)$ then by the Bernstein inequality (4) the formula (9) can be rewritten as

$$\|f\|^2 \le \frac{1+\alpha}{\alpha}\frac{\omega}{\Lambda(\Xi)}\|f\|^2 + (1+\alpha)\sum_j |\langle f, \xi_j\rangle|^2 .$$

If (15) holds then one has

$$0 < (1-\gamma)\|f\|^2 \le (1+\alpha)\sum_j |\langle f, \xi_j\rangle|^2 .$$

On the other hand, since

$$\left| \sum_{v\in V(\Omega_j)} f(v) \right|^2 \le |\Omega_j| \left(\sum_{v\in V(\Omega_j)} |f(v)|^2 \right),$$

one has

$$\sum_j |\,\langle f, \xi_j\rangle\,|^2 = \sum_j \frac{1}{|\Omega_j|}\left| \sum_{v\in V(\Omega_j)} f(v) \right|^2 \le$$

$$\sum_j \left(\sum_{v\in V(\Omega_j)} |f(v)|^2 \right) \le \|f\|^2.$$

Theorem is proven. ■

Note, that for the classical Paley-Wiener spaces on the real line the inequalities similar to (16) in the case when $\{\xi_j\}$ are delta functions were proved by Plancherel and Polya. Today they are better known as the frame inequalities.

Now we can formulate sampling theorem based on average values.

Theorem 5.2: Under the same conditions and notations as above every function $f \in PW_\omega(L)$ is uniquely determined by its averages $\langle f, \xi_j\rangle$ and can be reconstructed from this set of values in a stable way.

A. Reconstruction algorithms in terms of frames

What we just proved in the previous section is that under the same assumptions as above the set of functionals $f \to \langle f, \xi_j\rangle$ is a frame in the subspace $PW_\omega(L)$. This fact allows to apply the well known result of Duffin and Schaeffer which describes a stable method of reconstruction of a function $f \in PW_\omega(L)$ from a set of samples $\{\langle f, \xi_j\rangle\}$.

Theorem 5.3: If all the conditions of Theorem 4.1 are satisfied then there exists a dual frame $\{\theta_j\}$ in $PW_\omega(L)$ such that

$$f = \sum_j \langle f, \xi_j \rangle \theta_j = \sum_j \langle f, \theta_j \rangle \mathcal{P}\xi_j$$

where $\mathcal{P}$ is the orthogonal projection of $\ell^2(G)$ onto $PW_\omega(L)$.

Another posibility for reconstruction is to use the so-called frame algorithm [2].

VI. Average Variational Splines and a reconstruction algorithm

A. Variational interpolating splines

As in the previous sections we assume that G is a connected finite or infinite and countable graph and $\Xi = \{\Omega_j\}$ is a disjoint cover of $V(G)$ by connected and finite subgraphs Ω_j.

For a given sequence $\mathbf{v} = \{v_j\} \in l_2$ the set of all functions in $\ell^2(G)$ such that $\langle f, \xi_j \rangle = v_j$ will be denoted by $Z_{\mathbf{v}}$. In particular, $Z_{\mathbf{0}}$ corresponds to the sequence of zeros. We consider the following optimization problem:

For a given sequence $\mathbf{v} = \{v_j\} \in l_2$ find a function f in the set $Z_{\mathbf{v}} \subset \ell^2(G)$ which minimizes the functional

$$u \to \|L^{k/2}u\|, \quad u \in Z_{\mathbf{v}}.$$

Theorem 6.1: Under the above assumptions the optimization problem has a unique solution for every k.

Proof: Using Theorem 4.1 one can justify the following algorithm (see [9], [10]):

1) Pick any function $f \in Z_{\mathbf{v}}$.
2) Construct $P_0 f$ where P_0 is the orthogonal projection of f onto $Z_{\mathbf{0}}$ with respect to the inner product
$$\langle f, g \rangle_k = \sum_j \langle f, \xi_j \rangle \langle g, \xi_j \rangle + \langle L^{k/2} f, L^{k/2} g \rangle.$$
3) The function $f - P_0 f$ is the unique solution to the given optimization problem.

■

Definition 3: For $f \in \ell^2(G)$ the interpolating variational spline is denoted by $s_k(f)$ and it is the solution of the minimization problem such that $s_k(f) - f \in Z_{\mathbf{0}}$.

One can easily prove the following characterization of variational splines

Theorem 6.2: A function $u \in \ell^2(G)$ is a variational spline if and only if $L^k u$ is orthogonal to $L^k Z_{\mathbf{0}}$.

B. Reconstruction using splines

By using the same reasoning as in [9], [10] one can prove the following reconstruction theorem. Below we are keeping notations of Theorem 5.1.

Theorem 6.3: If the assumptions of Theorem 5.1 are satisfied then any function f in $PW_\omega(L)$, $\omega > 0$, can be reconstructed from a set of its averages $\{\langle f, \xi_j \rangle\}$ using the formula

$$f = \lim_{m \to \infty} s_m(f), \quad m = 2^l, \quad l = 0, 1, ...,$$

and the error estimate is

$$\|f - s_m(f)\| \le 2\gamma^m \|f\|, \quad m = 2^l, \quad l = 0, 1, ...,$$

where $\gamma < 1$.

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