A non-commutative viewpoint on graph signal processing

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***Abstract*—The emerging field of *graph signal processing* aims to develop analysis and processing techniques for data that is best represented on irregular domains such as graphs. To this end, important notions of classical signal processing, such as smoothness, band-limitedness, and sampling, should be extended to the case of graph signals. One of the most fundamental concepts in classical signal processing is the Fourier transform. Recently, graph Fourier transform was defined as a generalization of the Fourier transform on Abelian groups, and many of its properties were investigated. However, a graph is usually the manifestation of a non-commutative structure; this can be easily seen in the case of the Cayley graph of a non-Abelian group. In this article, we investigate a new approach to develop concepts of Fourier analysis for graphs. Our point of view is inspired by the theory of non-commutative harmonic analysis, and is founded upon the representation theory of non-Abelian groups.**

I. INTRODUCTION

Given the increasing amount of data being recorded as signals which are naturally represented on graph structures, the new field of graph signal processing has attracted the attention of many researchers in the past few years. For a fixed graph G, a graph signal on G is a complex-valued function $f : V \to \mathbb{C}$ on the vertex set V of G. If the set V is given a fixed ordering, say $\{v_i\}_{i=1}^N$, then the graph signal can be represented as a column vector $\big(f(v_1), f(v_2), \cdots, f(v_N)\big)^{\mathrm{T}}$ in $\mathbb{C}^N$. A major goal of graph signal processing is to analyze such signals not only as vectors in $\mathbb{R}^N$ or $\mathbb{C}^N$, but to take the underlying structure of the graph G into account. As a first step, we design orthonormal bases for $\mathbb{C}^N$ inspired from the graph G itself. Such an orthonormal basis can be designed as a set of eigenvectors for the graph adjacency matrix or the graph Laplacian. The adjacency matrix of G is a $0/1$-valued matrix A_G of size N, whose (i, j)-th entry is 1 precisely when the vertices v_i and v_j are adjacent. The Laplacian of G is an $N \times N$ matrix, denoted by L_G and defined as $L_G = D_G - A_G$, where D_G is the diagonal matrix with d_{ii} equal to the degree of the vertex v_i for every $i \in \{1, \ldots, N\}$.

Consider either of the matrices A_G or L_G, and fix an orthonormal basis of eigenvectors $\phi_1, \ldots, \phi_N \in \mathbb{R}^N$ associated with (possibly repeated) eigenvalues $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$ for that matrix. The *graph Fourier transform* $\widehat{f}$ of a graph signal $f : V \to \mathbb{C}$ is defined to be the expansion of f in terms of the orthonormal basis $\{\phi_i\}_{i=1}^N$. More precisely, we define

$$\widehat{f}(\phi_i) = \langle f, \phi_i \rangle = \sum_{n=1}^{N} f(v_n)\overline{\phi_i(v_n)}. \tag{I.1}$$

The corresponding *inverse Fourier transform* is given by

$$f(v_n) = \sum_{i=1}^{N} \widehat{f}(\phi_i)\phi_i(v_n), \tag{I.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{C}^N$. See [4], [5], [6] for detailed background on graph Fourier transform, and [3] for a general overview of graph signal processing.

The above definition of the graph Fourier transform is a direct generalization of the classical Fourier transform for vectors in $\mathbb{C}^N$. Every such vector can be identified with a complex-valued function on $\mathbb{Z}_N$. The space $\mathbb{Z}_N$ encodes a finite set of N elements, which admits a group structure as well (addition modulo N). To put the discussion in a harmonic analytic perspective, we think of $\mathbb{Z}_N$ as a compact group. As it is customary for compact groups, we equip $\mathbb{Z}_N$ with the normalized counting measure μ, *i.e.* $\mu(E) = \frac{|E|}{N}$ for every subset E of $\mathbb{Z}_N$ with $|E|$ distinct elements. The dual of the group $\mathbb{Z}_N$, which is denoted by $\widehat{\mathbb{Z}_N}$, is the group of characters χ_k with $k \in \{0, \ldots, N-1\}$, where each $\chi_k : \mathbb{Z}_N \to \mathbb{T}$ is defined by

$$\chi_k(m) = e^{\frac{2\pi i k m}{N}}, \text{ for } m \in \mathbb{Z}_N = \{0, 1, \ldots, N-1\}.$$

Clearly, $\big\{\chi_k : k = 0, \ldots, N-1\big\}$ forms an orthonormal basis of $\ell^2(\mathbb{Z}_N, \mu)$. The classical Fourier transform of a signal $f : \mathbb{Z}^N \to \mathbb{C}$ at $n \in \mathbb{Z}_N$ is defined by

$$\widehat{f}(n) = \langle f, \chi_n \rangle_{\ell^2(\mathbb{Z}_N)} = \frac{1}{N} \sum_{m=0}^{N-1} f(m)\overline{\chi_n(m)}. \tag{I.3}$$

The inverse Fourier transform can be written in a similar manner, once an appropriate translation-invariant measure is fixed for $\widehat{\mathbb{Z}_N}$. Guided by the theory of commutative harmonic analysis, we think of $\widehat{\mathbb{Z}_N}$ as a discrete group (as it is the dual of a compact group), and we equip it with the usual counting measure. The inverse Fourier transform then becomes

$$f(m)=\sum_{n=0}^{N-1}\widehat{f}(n)\chi_n(m), \quad \text{for } m\in\mathbb{Z}_N. \tag{I.4}$$

We refer the reader to [9] for an introduction to Fourier analysis on Abelian groups.

Let $\theta_N = e^{\frac{2\pi i}{N}}$ denote the first N-th root of unity. We naturally identify the function χ_k on $\mathbb{Z}_N$ with the column vector $\left(\theta_N^k, \theta_N^{2k}, \cdots, \theta_N^{(N-1)k}\right)^{\mathrm{T}}$ in $\mathbb{C}^N$, which we again denote by χ_k. The set of vectors $\left\{\frac{1}{\sqrt{N}}\chi_k : k=0,\ldots,N-1\right\}$ forms an orthonormal basis for $\mathbb{C}^N$. Applying formulas similar to (I.1) and (I.2), we obtain the graph Fourier transform from the classical formulas (I.3) and (I.4). We remark that the two transforms differ only in multiplicative factors, which arise from the different normalizations.

A natural question regarding graph Fourier transform is to what extent it resembles the actual group Fourier transform when the underlying graph is Cayley. We initiate a systematic study of this question. As a first step, we use representations of the group to construct suitable eigenbases for developing the Fourier transform of a Cayley graph (see Theorem III.1). Using these eigenbases simplifies several operations on graph signals including the graph translation operator, as we show in Theorem III.2. We conclude the paper by constructing a family of tight frames in Theorem III.5. Our frame construction is based on the results of Theorems III.1 and III.2.

II. Preliminaries

Throughout this section, we let G be a finite (not necessarily Abelian) group of size N. A (unitary) representation of G of dimension d is a group homomorphism $\pi : G \to \mathrm{U}_d(\mathbb{C})$, where $\mathrm{U}_d(\mathbb{C})$ is the (multiplicative) group of $d\times d$ unitary matrices. For a given representation π as above, a subspace W of $\mathbb{C}^d$ is called π-invariant if $\pi(g)\xi\in W$ for all $g\in G$ and all $\xi\in W$. A representation π is called irreducible if $\{0\}$ and $\mathbb{C}^d$ are its only π-invariant subspaces. Two representations π and σ of G are called (unitarily) equivalent, if there exists a unitary matrix U such that $U^{-1}\pi(g)U=\sigma(g)$ for all $g\in G$. We let $\widehat{G}$ denote the collection of all (equivalence classes of) irreducible unitary representations of G. In the case of an Abelian group, every irreducible representation of G is 1-dimensional [2, Corollary 3.6], and $\widehat{G}$ reduces to the group of characters on G.

Let $G=\{g_1,\ldots,g_N\}$ be a finite group. An important representation of G is the right regular representation $\rho : G\to \mathrm{U}_N(\mathbb{C})$, where $\rho(g)$ denotes the matrix associated with the permutation $h\mapsto hg$, $h\in G$. The representation ρ is not irreducible. Indeed, fix an irreducible representation $\pi\in\widehat{G}$ of dimension d_π and vectors $\xi,\eta\in\mathbb{C}^{d_\pi}$, and define the vector

$$\pi_{\xi,\eta}=\Big(\langle\pi(g_1)\xi,\eta\rangle,\ldots,\langle\pi(g_N)\xi,\eta\rangle\Big)^{\mathrm{T}}\in\mathbb{C}^N.$$

Vectors (or functions) of the form $\pi_{\xi,\eta}$ are called coefficient functions associated with π, and play a significant role in harmonic analysis of non-Abelian groups. For every $\eta\in\mathbb{C}^{d_\pi}$, the set $W_\eta=\left\{\pi_{\xi,\eta} : \xi\in\mathbb{C}^{d_\pi}\right\}$ forms a ρ-invariant subspace of $\mathbb{C}^N$. This fact is formalized in the well-known Peter–Weyl theorem, stating that ρ can be decomposed into a direct sum of irreducible representations. Moreover, every irreducible representation $\pi\in\widehat{G}$ occurs d_π many times in ρ, where d_π is the dimension of the representation π. We refer the reader to [2] for a detailed account of the representation theory of compact groups. We remark that everything we have mentioned so far in this section can be discussed in the context of general compact groups; however, for the purposes of this article, we limit ourselves to finite groups.

Let G be a finite group, and $S\subseteq G$ be a generating set for G. Assume in addition that S is inverse-closed, *i.e.*, $x^{-1}\in S$ for all $x\in S$. The Cayley graph $\Gamma(G;S)$ is the graph with vertex set G in which a pair of vertices x and y are adjacent if and only if $x^{-1}y\in S$. Note that the assumption of S being inverse-closed guarantees that the Cayley graph is undirected. Observe that the neighborhood of a vertex x in $\Gamma(G;S)$ is given by

$$xS := \{xs \;:\; s\in S\}.$$

As a result, the adjacency matrix A of the Cayley graph $\Gamma(G;S)$ is given by

$$A=\sum_{s\in S}\rho(s),$$

where ρ is the right regular representation of G.

III. Graph Fourier transform for Cayley graphs

Let G be a finite group of size N. Recall that by the Peter–Weyl theorem, the right regular representation ρ is unitarily equivalent to $\bigoplus_{\pi\in\widehat{G}} d_\pi\pi$, *i.e.* there exists a unitary matrix U so that

$$\rho(g)=U^*\Big(\bigoplus_{\pi\in\widehat{G}} d_\pi\pi(g)\Big)U,\ \forall g\in G.$$

Now, for $\pi\in\widehat{G}$, let $\{e_1,\ldots,e_{d_\pi}\}$ be the standard basis of $\mathbb{C}^{d_\pi}$, and define $\pi_{i,j}$ to be the coefficient function π_{e_j,e_i}. Under the application of the unitary map U, we can think of $\pi_{i,j}$ as an element of $\mathbb{C}^N$ (or, equivalently, as an element of $\ell^2(G)$) whose projection onto every summand of $\bigoplus_{\pi\in\widehat{G}} d_\pi\mathbb{C}^{d_\pi}$, except the summand associated with the j-th copy of π, is 0.

The following theorem follows from Theorem 6 in [1], and suggests a suitable eigenbasis to be used in the Fourier expansion of a signal defined on a Cayley graph. It was proven for Cayley graphs as Theorem 1.1 in [8]. We include a simplified proof to be self-contained and illustrate the differences of defining Cayley graphs through left, rather than right, cosets.

Theorem III.1. *Let G be a finite group and S be an inverse-closed generating set in G which can be written as a union of some conjugacy classes of G. Let A be the adjacency matrix of the Cayley graph $\Gamma(G;S)$, and let π and $\pi_{i,j}$ be as above. Then*

$$A\pi_{i,j}=\lambda_\pi\pi_{i,j},$$

where $\lambda_\pi = \frac{1}{d_\pi}\sum_{g\in S}\mathrm{Tr}(\pi(g))$.

Proof. First let $\mathbb{1}_S$ be the indicator function of S and recall that

$$A = \sum_{s\in S}\rho(s) = \sum_{g\in G}\mathbb{1}_S(g)\rho(g).$$

Define $\mathcal{E}_{\pi,i} := \mathrm{span}\{\pi_{i,j} : 1 \le j \le d_\pi\}$. Then by Peter–Weyl, $\mathcal{E}_{\pi,i}$ is an invariant subspace of ρ such that, for U restricted to this subspace, $U^*\pi(g)U = \rho^{\mathcal{E}_{\pi,i}}(g)$ for all $g \in G$, where $\rho^{\mathcal{E}_{\pi,i}}(\cdot)$ denotes the restriction of the operator $\rho(\cdot)$ to $\mathcal{E}_{\pi,i}$. Hence,

$$\begin{aligned}
A\pi_{i,j} &= \sum_{g\in G}\mathbb{1}_S(g)\rho^{\mathcal{E}_{\pi,i}}(g)\pi_{i,j} = \sum_{g\in G}\mathbb{1}_S(g)U^*\pi(g)U\pi_{i,j} \\
&= \sum_{g\in G}\mathbb{1}_S(g)U^*\pi(g)e_j = \sum_{k=1}^{d_\pi}\sum_{g\in G}\mathbb{1}_S(g)\pi_{k,j}(g)\pi_{i,k} \\
&= \left[\frac{1}{d_\pi}\sum_{g\in S}\chi_\pi(g)\right]\pi_{i,j},
\end{aligned}$$

where the final equality follows from Schur's lemma ([2], Theorem 3.5) and the fact that $\mathbb{1}_S$ is a class function when S is the union of conjugacy classes. Here $\chi_\pi(g)$ is the trace of the matrix $\pi(g)$. □

The above eigenbasis is natural to analyze signals defined on Cayley graphs. It can be used to simplify many complications arising in signal processing on such graphs, and results in signal processing that is best compatible with the underlying structure of the graph. For example, let us examine the graph translation operator defined in [6] via a convolution with the Kronecker delta function δ_m, following the traditional Abelian definition:

$$(T_m f)(v_n) = \sqrt{N}(f*\delta_m)(v_n) = \sqrt{N}\sum_{i=0}^{N-1}\widehat{f}(\phi_i)\overline{\phi_i(v_m)}\phi_i(v_n),$$

where $\{\phi_i\}_{i=0}^{N-1}$ is the eigenbasis for the graph Fourier transform.

Theorem III.2. *Let G be a finite group of size N, and consider the Cayley graph $\Gamma(G;S)$, where $S = \bigcup_{i=1}^s C_i$ is a union of conjugacy classes $C_1,\dots,C_s$ in G. Let g be a signal on $\Gamma(G;S)$, and suppose that for every $\pi\in\widehat{G}$, $\widehat{g}(\pi_{i,j}) := \widehat{g}(\pi)$ attains the same value for all $1\le i,j\le d_\pi$. Then the graph translation operator is given by*

$$(T_\ell g)(v_k) = \frac{1}{\sqrt{N}}\sum_{\pi\in\widehat{G}}d_\pi\widehat{g}(\pi)\chi_\pi(\ell^{-1}k).$$

Proof. By Theorem III.1, the collection $\bigcup_{\pi\in\widehat{G}}\{\pi_{i,j}\}_{i,j=1}^{d_\pi}$ forms an orthogonal basis for $\mathbb{C}^N$. To obtain an orthonormal basis, we normalize these vectors as $\sqrt{\frac{d_\pi}{|G|}}\pi_{i,j}$. Applying the definition of $T_\ell g$, we obtain:

$$(T_\ell g)(v_k) = \frac{\sqrt{N}}{|G|}\sum_{\pi\in\widehat{G}}\sum_j\sum_i\widehat{g}(\pi_{i,j})\overline{\pi_{i,j}}(\ell)\pi_{i,j}(k),$$

which simplifies as follows:

$$\begin{aligned}
(T_\ell g)(v_k) &= \frac{\sqrt{N}}{|G|}\sum_{\pi\in\widehat{G}}d_\pi\widehat{g}(\pi)\sum_j\sum_i\overline{\pi}_{i,j}(\ell)\pi_{i,j}(k) \\
&= \frac{\sqrt{N}}{N}\sum_{\pi\in\widehat{G}}d_\pi\widehat{g}(\pi)\sum_j\sum_i\pi_{j,i}(\ell^{-1})\pi_{i,j}(k) \\
&= \frac{1}{\sqrt{N}}\sum_{\pi\in\widehat{G}}d_\pi\widehat{g}(\pi)\sum_j[\pi(\ell^{-1}k)]_{j,j} \\
&= \frac{1}{\sqrt{N}}\sum_{\pi\in\widehat{G}}d_\pi\widehat{g}(\pi)\chi_\pi(\ell^{-1}k).
\end{aligned}$$

□

We remark that the above proof depends heavily on the assumption that $\widehat{g}$ depends only on π, i.e., is constant on $\pi_{i,j}$ for different values of i, j.

Corollary III.3. *Under the assumptions of Theorem III.2, the translation operator T_ℓ is invariant when shifted in both indices, that is, for all $m\in G$,*

$$(T_\ell g)(v_k) = (T_{\ell m}g)(v_{km}) = (T_{m\ell}g)(v_{mk}).$$

In particular, choosing $m = \ell^{-1}$, we see that

$$(T_\ell g)(v_k) = (T_e g)(v_{\ell^{-1}k}),$$

where e is the group identity.

Proof. By Theorem III.2, we have

$$\begin{aligned}
(T_{\ell m}g)(v_{km}) &= \frac{1}{\sqrt{N}}\sum_{\pi\in\widehat{G}}d_\pi\widehat{g}(\pi)\chi_\pi((\ell m)^{-1}km) \\
&= \frac{1}{\sqrt{N}}\sum_{\pi\in\widehat{G}}d_\pi\widehat{g}(\pi)\chi_\pi(m^{-1}\ell^{-1}km) \\
&= \frac{1}{\sqrt{N}}\sum_{\pi\in\widehat{G}}d_\pi\widehat{g}(\pi)\chi_\pi(\ell^{-1}k) \\
&= (T_\ell g)(v_k),
\end{aligned}$$

where the last equality follows from the fact that characters are class functions. The equality $(T_\ell g)(v_k) = (T_{m\ell}g)(v_{mk})$ is proved similarly. □

Corollary III.4. *Under the assumptions of Theorem III.2, the translation operator T_e is invariant on the conjugacy classes of G.*

Proof. From Corollary III.3, we have

$$(T_e g)(v_k) = (T_{\ell e\ell^{-1}}g)(v_{\ell k\ell^{-1}}) = (T_e g)(v_{\ell k\ell^{-1}}) \quad \forall \ell\in G.$$

□

We can now use the results obtained so far to build many localized functions from the characters of the underlying group, although we will demonstrate in future work that not all localized functions can be constructed in this manner.

Theorem III.5. *Let $\{s_1, s_2, \cdots, s_k\}$ be a complete set of representatives of the conjugacy classes of a group G. Consider the Cayley graph $\Gamma(G;S)$, where S is the union of some conjugacy classes. Define $K_r := d(v_e, v_{s_r})$, with d being the graph distance function. For each s_r, define $g_{(s_r)}$ in the spectral domain by*

$$\widehat{g_{(s_r)}}(\pi_{i,j}) := \widehat{g_{(s_r)}}(\pi) = \frac{\overline{\chi_\pi(s_r)}}{d_\pi}.$$

Then for each $i = 1, 2, \cdots, N$, and each $r = 0, 1, 2, \cdots, k$, $T_i g_{(s_r)}$ is supported on the K_r-ball centered at vertex v_i. Moreover, these functions form a basis for localized functions whose graph Fourier tranform is constant on each representation space.

Proof. Let $C(h)$ be the conjugacy class of the element $h \in G$. Then for g as defined above, we have by theorem III.2

$$(T_\ell g)(v_k) = (T_e g)(v_{\ell^{-1}}) = \frac{1}{N^{\frac{1}{2}}} \sum_{\pi \in \widehat{G}} d_\pi \widehat{g}(\pi) \chi_\pi(\ell^{-1}k)$$

$$= \frac{1}{N^{\frac{1}{2}}} \sum_{\pi \in \widehat{G}} d_\pi \frac{\overline{\chi_\pi(s_i)}}{d_\pi} \chi_\pi(\ell^{-1}k) = \frac{1}{N^{\frac{1}{2}}} \sum_{\pi \in \widehat{G}} \overline{\chi_\pi(s_i)} \chi_\pi(\ell^{-1}k).$$

Thinking of the last sum as an inner product in $\ell^2(\widehat{G})$, we conclude that

$$(T_\ell g)(v_k) = \begin{cases} 0 & \text{if } \ell^{-1}k \notin C(s_i) \\ \frac{|G|}{|C(s_i)|} & \text{if } \ell^{-1}k \in C(s_i) \end{cases},$$

where we used the fact that the columns of character tables are orthogonal.

Now suppose $\ell, k \in G$ are such that the graph distance $d(e, \ell^{-1}k) = d(\ell, k) > K_i = d(e, s_i)$. Then as distances are constant on conjugacy classes by corollary III.3, we obtain $(T_\ell g)(v_k) = 0$ as $\ell^{-1}k \notin C(s_i)$.

To see that these functions form a basis, we simply note that the lifted characters $\left\{\widehat{g}_{(s_r)}\right\}_{r=1}^{k}$ forms a basis for the subspace of functions whose Fourier transforms are constant on representations, as they are linearly independent and there are $|\widehat{G}|$ many of them. For any graph distance K_r then, it is clear from above that the functions $g_{(s_i)}$ for which $d(e, s_i) < K_r$ are all localized, and those for which $d(e, s_i) \geq K_r$ are not. They thus form a basis for such functions. ☐

We conclude this paper by constructing a family of tight frames. Recall that the graph signal modulation operator M_k is given by

$$(M_k f)(v_n) = \sqrt{N} f(v_n) \phi_k(v_n) \qquad (k = 0, 1, \cdots, N-1),$$

where $\{\phi_i\}$ is the graph Fourier basis (see [7] for more details on frames, graph signal processing, and related operators).

Theorem III.6. *Consider a Cayley graph $\Gamma(G;S)$, where S is some union of conjugacy classes. Let the graph Fourier basis be the eigenvectors for the graph adjacency matrix (or the graph Laplacian) formed by normalizing the coefficient functions of the irreducible unitary representations of G as in Theorem III.1. Then for window functions $\widehat{\mathfrak{g}}(\pi) := \sum_k \alpha_k \frac{\overline{\chi_\pi(g_k)}}{d_\pi}$ (with each g_k in a distinct conjugacy class) obtained from lifting characters (Theorem III.5), the $|G|^2$ vectors $\mathfrak{g}_{g,\pi_{i,j}} := M_{\pi_{i,j}}(T_g \mathfrak{g})$ ($g \in G, \pi \in \widehat{G}, 1 \leq i,j \leq d_\pi$) form a tight frame for graph signals f on $\Gamma(G;S)$. Furthermore, $\|T_e g\| = \sum_k \frac{|\alpha_k|^2}{|c(g_k)|}$.*

Proof. We have

$$\begin{aligned}
&\sum_{g \in G} \sum_{\pi \in \widehat{G}} \sum_{i=1}^{d_\pi} \sum_{j=1}^{d_\pi} |\langle f, \mathfrak{g}_{g,\pi_{i,j}} \rangle|^2 \\
=\;& \sum_{g \in G} \sum_{\pi \in \widehat{G}} \sum_{i=1}^{d_\pi} \sum_{j=1}^{d_\pi} |\langle f, M_{\pi_{i,j}} T_g \mathfrak{g} \rangle|^2 \\
=\;& \sum_{g \in G} \sum_{\pi \in \widehat{G}} \sum_{i=1}^{d_\pi} \sum_{j=1}^{d_\pi} |\langle f, \sqrt{|G|} \sqrt{\frac{d_\pi}{|G|}} \pi_{i,j} \circ (T_g \mathfrak{g}) \rangle|^2 \\
=\;& |G| \sum_{g \in G} \sum_{\pi \in \widehat{G}} \sum_{i=1}^{d_\pi} \sum_{j=1}^{d_\pi} |\langle f \circ (\overline{T_g \mathfrak{g}}), \sqrt{\frac{d_\pi}{|G|}} \pi_{i,j} \rangle|^2 \\
=\;& |G| \sum_{g \in G} \|\widehat{f \circ (\overline{T_g \mathfrak{g}})}\|_2^2 = |G| \sum_{g \in G} \|f \circ (\overline{T_g \mathfrak{g}})\|_2^2 \\
=\;& |G| \sum_{g \in G} \sum_{h \in G} |f \circ (\overline{T_g \mathfrak{g}})(h)|^2 \\
=\;& |G| \sum_{h \in G} |f(h)|^2 \|T_e \mathfrak{g}\|_2^2 \\
=\;& |G| \|T_e \mathfrak{g}\|_2^2 \|f\|_2^2,
\end{aligned}$$

where $\circ$ represents Hadamard (entry-wise) multiplication of the vectors. To finish the proof, it is enough to show that $\|T_e \mathfrak{g}\|_2^2 = \sum_k \frac{|\alpha_k|^2}{|c(g_k)|}$, whose details we skip. ☐

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