A directional periodic uncertainty principle

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***Abstract*—A notion of a directional uncertainty product (UP) for multivariate periodic functions is introduced. It is a characteristic of a localization for a signal along a fixed direction.**

I. Introduction

A notion of uncertainty product (UP) is a sufficiently well-studied object in harmonic analysis. Initially, it was introduced for functions on the real line to measure a simultaneous localization of a function and its Fourier transform [1]. The essence of this quantification of localization is contained in the Heisenberg uncertainty principle, which says that for any appropriate function the UP cannot be smaller than a positive absolute constant. Later, numerous versions of this general principle were developed for different algebraic and topological structures such as abstract locally compact groups, high-dimensional spheres, etc. (see, e.g., [2], [3], [4]). For more detailed information concerning this topic, we refer the interested reader to surveys [5] and [6] and the references therein.

In this paper we focus on the case of multivariate periodic functions and multivariate discrete signals. For periodic functions of one variable a notion of UP was introduced in 1985 by Breitenberger in [7]. The corresponding uncertainty principle is also valid in this setup. One possible extension of this notion to the case of multivariate periodic functions was suggested by Goh and Goodman in [8] (see formula (2)). However, this approach does not take into account the main difference between periodic functions of one variable and many variables, namely the localization of a function along particular directions. The main contribution of this paper is a new approach that allows to include the directionality into the definition of the UP (see formula (3)). We compare these two approaches and and discuss the differences in detail (see also [9]). At the same time, both definitions fit into a more general operator approach (see formula (1)). This approach was established by Folland in [10] and was extended to two normal or symmetric operators by Selig in [11] and Goh, Micchelli in [12]. For several operators this approach was generalized by Goh and Goodman in [8].

II. Basic notations and definitions

We use the standard multi-index notation. Let $d \in \mathbb{N}$, $\mathbb{R}^d$ be the d-dimensional Euclidean space, $\{e_j, 1 \leq j \leq d\}$ be the standard basis in $\mathbb{R}^d$, $\mathbb{Z}^d$ be the integer lattice in $\mathbb{R}^d$, $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ be the d-dimensional torus. Let $x = (x_1, \dots, x_d)^{\mathrm{T}}$ and $y = (y_1, \dots, y_d)^{\mathrm{T}}$ be column vectors in $\mathbb{R}^d$. Then $\langle x, y \rangle := x_1 y_1 + \dots + x_d y_d$, $\|x\| := \sqrt{\langle x, x \rangle}$, $\|x\|_1 = \sum_{j=1}^d |x_j|$, $\|x\|_\infty = \max_j |x_j|$. We say that $x \geq y$, if $x_j \geq y_j$ for all $j = 1, \dots, d$, and we say that $x > y$, if $x \geq y$ and $x \neq y$. Further, $\mathbb{Z}_+^d := \{\alpha \in \mathbb{Z}^d : \ \alpha \geq \ \mathbf{0}\}$, where $\mathbf{0} = (0, \dots, 0)$ denotes the origin in $\mathbb{R}^d$. For $\alpha = (\alpha_1, \dots, \alpha_d)^{\mathrm{T}} \in \mathbb{Z}_+^d$, denote $|\alpha| := \alpha_1 + \dots + \alpha_d$. For $x \in \mathbb{R}$, $x_+ := \begin{cases} 0, & x \leq 0, \\ x, & x > 0. \end{cases}$

For a sufficiently smooth function f defined on $\Omega \subset \mathbb{R}^d$ and a multi-index $\alpha \in \mathbb{Z}_+^d$, $D^\alpha f$ denotes the derivative of f of order α and $D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x^\alpha} = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$. For $\alpha = e_j$, we also use $D^{e_j} f = f_j'$. The directional derivative of a sufficiently smooth function f defined on Ω along a vector $L = (L_1, ..., L_d) \in \mathbb{R}^d$ is denoted by $\frac{\partial f}{\partial L} = \sum_{j=1}^d L_j \frac{\partial f}{\partial x_j}$.

For a function $f \in L_2(\mathbb{T}^d)$ its norm is denoted by $\|f\|_{\mathbb{T}^d}^2 = \int_{\mathbb{T}^d} |f(x)|^2 \mathrm{d}x$. The Fourier series coefficients of a function $f \in L_2(\mathbb{T}^d)$ are given by $c_k = c_k(f) = \widehat{f}(k) = \int_{\mathbb{T}^d} f(x) \mathrm{e}^{-2\pi \mathrm{i} \langle k, x \rangle} \mathrm{d}x$, $k \in \mathbb{Z}^d$. The Sobolev space $H^1(\mathbb{T}^d)$ consists of functions in $L_2(\mathbb{T}^d)$ such that all its derivatives of the first order are also in $L_2(\mathbb{T}^d)$, which can be written as

$$H^1(\mathbb{T}^d) = \left\{ f \in L_2(\mathbb{T}^d) : \sum_{k \in \mathbb{Z}^d} \|k\|^2 |c_k(f)|^2 < \infty \right\}.$$

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and with norm $\| \cdot \| := \langle \cdot, \cdot \rangle^{1/2}$. Let $\mathcal{A}$, $\mathcal{B}$ be two linear operators with domains $\mathcal{D}(\mathcal{A})$, $\mathcal{D}(\mathcal{B}) \subseteq \mathcal{H}$ and ranges in $\mathcal{H}$. The variance of non-zero $f \in \mathcal{D}(\mathcal{A})$ with respect to the operator $\mathcal{A}$ is defined to be

$$\Delta(\mathcal{A}, f) = \|\mathcal{A} f\|^2 - \frac{|\langle \mathcal{A} f, f \rangle|^2}{\|f\|^2} = \left\| \left(\mathcal{A} - \frac{\langle \mathcal{A} f, f \rangle}{\|f\|^2} \right) f \right\|^2$$

$$= \min_{\alpha \in \mathbb{C}} \|\mathcal{A} f - \alpha f\|^2.$$

We recall that a densely defined linear operator $\mathcal{A}$ in a Hilbert space $\mathcal{H}$ is said to be symmetric if $\langle \mathcal{A} f, g \rangle = \langle f, \mathcal{A} g \rangle$ for $f, g \in \mathcal{D}(\mathcal{A})$. If additionally $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}^*)$, where $\mathcal{A}^*$ is an adjoint operator for $\mathcal{A}$, then $\mathcal{A}$ is self-adjoint. We say that $\mathcal{A}$ is normal if $\mathcal{A}$ is closed, densely defined and if $\mathcal{A}^* \mathcal{A} = \mathcal{A} \mathcal{A}^*$. For a normal operator $\mathcal{A}$ we have that $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}^*)$ and $\|\mathcal{A} f\| = \|\mathcal{A}^* f\|$ for any $f \in \mathcal{D}(\mathcal{A})$.

The commutator of $\mathcal{A}$ and $\mathcal{B}$ is defined by $[\mathcal{A}, \mathcal{B}] := \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$ with domain $\mathcal{D}(\mathcal{A}\mathcal{B}) \bigcap \mathcal{D}(\mathcal{B}\mathcal{A})$.

Theorem 1: [8, Theorem 4.1] Let $\mathcal{A}_1, \dots \mathcal{A}_n$, $\mathcal{B}_1, \dots \mathcal{B}_n$ be symmetric or normal operators acting from a Hilbert space $\mathcal{H}$ into itself. Then for any non-zero f in $\mathcal{D}(\mathcal{A}_j\mathcal{B}_j) \bigcap \mathcal{D}(\mathcal{B}_j\mathcal{A}_j)$, $j = 1, \dots, n$,

$$\frac{1}{4}\left(\sum_{j=1}^{n} |\langle[\mathcal{A}_j, \mathcal{B}_j]f, f\rangle|\right)^2 \leq \sum_{j=1}^{n} \Delta(\mathcal{A}_j, f) \sum_{j=1}^{n} \Delta(\mathcal{B}_j, f). \tag{1}$$

We recall that for two operators $\mathcal{A}$, $\mathcal{B}$ on a Hilbert space $\mathcal{H}$, which are symmetric or normal, the uncertainty principle was established by Selig in [11, Theorem 3.1] as

$$\frac{1}{4}|\langle[\mathcal{A}, \mathcal{B}]f, f\rangle|^2 \leq \Delta_{\mathcal{A}}(f)\Delta_{\mathcal{B}}(f)$$

for all non-zero $f \in \mathcal{D}(\mathcal{A}\mathcal{B}) \bigcap \mathcal{D}(\mathcal{B}\mathcal{A})$.

If the commutator $\langle[\mathcal{A}_j, \mathcal{B}_j]f, f\rangle$ is non-zero for all $j = 1, \dots, n$, then the UP for f is defined as

$$\mathrm{UP}(f) := \frac{\left(\sum_{j=1}^{n} \Delta(\mathcal{A}_j, f)\right)\left(\sum_{j=1}^{n} \Delta(\mathcal{B}_j, f)\right)}{\left(\sum_{j=1}^{n} |\langle[\mathcal{A}_j, \mathcal{B}_j]f, f\rangle|\right)^2}.$$

In this terms, the uncertainty principle says that $\mathrm{UP}(f)$ cannot be smaller than $\frac{1}{4}$, for any appropriate function f.

The well-known Heisenberg UP for functions defined on the real line fits in this operator approach, if $n = 1$, $\mathcal{H} = L_2(\mathbb{R})$ and the two operators are as follows $\mathcal{A}f(x) = 2\pi x f(x)$, $\mathcal{B}f(x) = \frac{\mathrm{i}}{2\pi}\frac{\mathrm{d}f}{\mathrm{d}x}(x)$.

The Breitenberger UP is defined for periodic functions. In this case, $n = 1$, $\mathcal{H} = L_2(\mathbb{T})$ and $\mathcal{A}^{\mathbb{T}}f(x) = \mathrm{e}^{2\pi \mathrm{i} x}f(x)$, $\mathcal{B}^{\mathbb{T}}f(x) = \frac{\mathrm{i}}{2\pi}\frac{\mathrm{d}f}{\mathrm{d}x}(x)$. The commutator is $[\mathcal{A}^{\mathbb{T}}, \mathcal{B}^{\mathbb{T}}] = \mathcal{A}^{\mathbb{T}}$. It is more convenient for the Breitenberger UP to use the notions of the angular and frequency variance. Since $\|\mathcal{A}^{\mathbb{T}}f\|^2_{\mathbb{T}} = \|f\|^2_{\mathbb{T}}$,

$$\mathrm{var}^A(f) := \frac{\|f\|^2_{\mathbb{T}}\Delta(\mathcal{A}^{\mathbb{T}}, f)}{|\langle[\mathcal{A}^{\mathbb{T}}, \mathcal{B}^{\mathbb{T}}]f, f\rangle|^2} = \left(\frac{\|f\|^2_{\mathbb{T}}}{|\langle\mathcal{A}^{\mathbb{T}}f, f\rangle|}\right)^2 - 1,$$

$$\mathrm{var}^F(f) := \frac{\Delta(\mathcal{B}^{\mathbb{T}}, f)}{\|f\|^2_{\mathbb{T}}} = \frac{\|\mathcal{B}^{\mathbb{T}}f\|^2_{\mathbb{T}}}{\|f\|^2_{\mathbb{T}}} - \frac{|\langle\mathcal{B}^{\mathbb{T}}f, f\rangle|^2}{\|f\|^4_{\mathbb{T}}},$$

$$\mathrm{UP}^{\mathbb{T}}(f) := \mathrm{var}^A(f)\mathrm{var}^F(f).$$

It is known that the lower bound for $\mathrm{UP}^{\mathbb{T}}$ is not attained at any function. But there exist sequences of functions such that $\mathrm{UP}^{\mathbb{T}}$ tends to the optimal value $\frac{1}{4}$ (see, e.g., [13]).

III. Main results

For the space $L_2(\mathbb{T}^d)$ of multivariate periodic functions, Goh and Goodman in [8] suggest to take the operators as follows $\mathcal{A}_j f(x) = \mathrm{e}^{2\pi \mathrm{i} x_j}f(x)$, $\mathcal{B}_j f(x) = \frac{\mathrm{i}}{2\pi}\frac{\partial f}{\partial x_j}(x)$, $j = 1, \dots, d$. Note that the domains of the operators are $\bigcap_{j=1}^{d} \mathcal{D}(\mathcal{A}_j) = L_2(\mathbb{T}^d)$, $\bigcap_{j=1}^{d} \mathcal{D}(\mathcal{B}_j) = H^1(\mathbb{T}^d)$. Operators $\mathcal{A}_j$ are normal, $\mathcal{B}_j$ are self-adjoint. The commutators for $f \in H^1(\mathbb{T}^d)$ are $[\mathcal{A}_j, \mathcal{B}_j]f = \mathcal{A}_j f$. The uncertainty principle for these operators is stated as follows.

Theorem 2: For a function $f \in H^1(\mathbb{T}^d)$, such that $\langle\mathcal{A}_j f, f\rangle \neq 0$ for some $j = 1, \dots, d$, the functional $\mathrm{UP}^{\mathbb{T}^d}_{GG}(f)$ is well-defined and

$$\mathrm{UP}^{\mathbb{T}^d}_{GG}(f) = \frac{\sum_{j=1}^{d}\left(\|f\|^4_{\mathbb{T}^d} - \left|\sum_{k\in\mathbb{Z}^d} c_{k-e_j}\overline{c_k}\right|^2\right)}{\left(\sum_{j=1}^{d}\left|\sum_{k\in\mathbb{Z}^d} c_{k-e_j}\overline{c_k}\right|\right)^2}$$

$$\sum_{j=1}^{d}\left(\frac{\sum_{k\in\mathbb{Z}^d} k_j^2|c_k|^2}{\|f\|^2_{\mathbb{T}^d}} - \left(\frac{\sum_{k\in\mathbb{Z}^d} k_j|c_k|^2}{\|f\|^2_{\mathbb{T}^d}}\right)^2\right) \geq \frac{1}{4}, \tag{2}$$

where $k = (k_1, \dots, k_d)$, $c_k = c_k(f)$ are the Fourier coefficients of f.

Defining the variances for $f \in H^1(\mathbb{T}^d)$ as

$$\mathrm{var}^A_{GG}(f) = \frac{\|f\|^2_{\mathbb{T}^d}\sum_{j=1}^{d}\Delta(\mathcal{A}_j, f)}{\left(\sum_{j=1}^{d}|\langle[\mathcal{A}_j, \mathcal{B}_j]f, f\rangle|\right)^2},$$

$$\mathrm{var}^F_{GG}(f) = \sum_{j=1}^{d}\Delta(\mathcal{B}_j, f)/\|f\|^2_{\mathbb{T}^d},$$

it can be shown, that the variances attain the value ∞ if and only if $\langle\mathcal{A}_j f, f\rangle = 0$, for all $j = 1, \dots, d$. In these cases, we can also assign to $\mathrm{UP}^{\mathbb{T}^d}_{GG}(f)$ the value ∞, except the following case $\mathrm{var}^F_{GG}(f) = 0$ and $\mathrm{var}^A_{GG}(f) = \infty$. This case happens if and only if f is a monomial. Indeed, $\mathrm{var}^F_{GG}(f) = 0$ if and only if $\Delta(\mathcal{B}_j, f) = 0$ for all $j = 0, \dots, d$, than implies $\mathcal{B}_j f = \beta_j f$ for some $\beta_j \in \mathbb{C}$, that is $\frac{\partial f}{\partial x_j} = \alpha_j f$ for some $\alpha_j \in \mathbb{C}$. Since $f \in H^1(\mathbb{T}^d)$, it follows that f is a monomial. However, in this case, i.e., $\mathrm{var}^F_{GG}(f) = 0$ and $\mathrm{var}^A_{GG}(f) = \infty$, inequality (1) takes the form $1/4 \cdot 0 \leq C \cdot 0$. It is trivially true. Thus, inequality (1) is valid for all non-zero functions $f \in H^1(\mathbb{T}^d)$.

In fact, the above approach for the definition of the UP does not deal with a new phenomenon, that appears in the multidimensional case, namely, the localization of a function along particular directions. We suggest an approach that allows to include the directionality into the definition.

The directional UP for $\mathbb{T}^d$ along a direction $L \in \mathbb{Z}^d$ ($L \neq \mathbf{0}$) is defined using the operators

$$\mathcal{A}_L f(x) = \mathrm{e}^{2\pi \mathrm{i}\langle L, x\rangle}f(x), \quad \mathcal{B}_L f(x) = \frac{\mathrm{i}}{2\pi}\frac{\partial f}{\partial L}(x).$$

with domains $\mathcal{D}(\mathcal{A}_L) = L_2(\mathbb{T}^d)$, $\mathcal{D}(\mathcal{B}_L) = H^1(\mathbb{T}^d)$. Note that $\mathcal{A}_L$ is normal, $\mathcal{B}_L$ is symmetric. The commutator for $f \in \mathcal{D}(\mathcal{A}_L) \cap \mathcal{D}(\mathcal{B}_L)$ is $[\mathcal{A}_L, \mathcal{B}_L]f = \|L\|^2\mathcal{A}_L f$. Thus, the directional UP for a function $f \in \mathcal{D}(\mathcal{A}_L) \cap \mathcal{D}(\mathcal{B}_L)$ such that $\mathcal{A}_L f \neq 0$ is defined as

$$\mathrm{UP}^{\mathbb{T}^d}_L(f) = \frac{1}{\|L\|^4_2}\left(\frac{\|f\|^4_{\mathbb{T}^d}}{|\langle\mathcal{A}_L f, f\rangle|^2} - 1\right)\left(\frac{\|\mathcal{B}_L f\|^2_{\mathbb{T}^d}}{\|f\|^2_{\mathbb{T}^d}} - \frac{|\langle\mathcal{B}_L f, f\rangle|^2}{\|f\|^4_{\mathbb{T}^d}}\right)$$

$$:= \frac{1}{\|L\|^4}\mathrm{var}^A_L(f)\mathrm{var}^F_L(f),$$

where $\mathrm{var}_L^A(f)$ is the angular directional variance and $\mathrm{var}_L^F(f)$ is the frequency directional variance.

Theorem 3: For $L\in\mathbb{Z}^d$ and a function $f\in H^1(\mathbb{T}^d)$, such that $\langle \mathcal{A}_L f, f\rangle \neq 0$, the functional $\mathrm{UP}_L^{\mathbb{T}^d}(f)$ is well-defined and

$$\mathrm{UP}_L^{\mathbb{T}^d}(f)=\frac{1}{\|L\|^4}\left(\frac{\left(\sum\limits_{k\in\mathbb{Z}^d}|c_k|^2\right)^2}{\left|\sum\limits_{k\in\mathbb{Z}^d}c_{k-L}\overline{c_k}\right|^2}-1\right)$$

$$\left(\frac{\sum\limits_{k\in\mathbb{Z}^d}\langle L,k\rangle^2|c_k|^2}{\sum\limits_{k\in\mathbb{Z}^d}|c_k|^2}-\left(\frac{\sum\limits_{k\in\mathbb{Z}^d}\langle L,k\rangle|c_k|^2}{\sum\limits_{k\in\mathbb{Z}^d}|c_k|^2}\right)^2\right)\geq\frac{1}{4}, \qquad (3)$$

where $c_k=c_k(f)$ are the Fourier coefficients of f.

The statement easily follows from the operator approach and

$$\mathcal{A}_L f(x)=\sum_{k\in\mathbb{Z}^d}c_{k-L}\mathrm{e}^{2\pi\mathrm{i}\langle k,x\rangle},$$

$$\mathcal{B}_L f(x)=-\sum_{k\in\mathbb{Z}^d}\langle L,k\rangle c_k\mathrm{e}^{2\pi\mathrm{i}\langle k,x\rangle}.$$

It can be shown, that the directional variances attain the value ∞ if and only if $\langle\mathcal{A}_L f,f\rangle=0$. In this case, we can also assign to $\mathrm{UP}_L^{\mathbb{T}^d}(f)$ the value ∞, except the following case $\mathrm{var}_L^F(f)=0$ and $\mathrm{var}_L^A(f)=\infty$. However in this case, (1) is trivially satisfied ($0\leq 0$), so inequality (1) is valid for operators $\mathcal{A}_L$ and $\mathcal{B}_L$ for all non-zero functions $f\in H^1(\mathbb{T}^d)$.

In contrast to the Breitenberger UP and to the UP defined by Goh and Goodman, the optimal function exists for the directional UP. Indeed, let $a(f)=\frac{\langle\mathcal{A}_L f,f\rangle}{\|f\|_2^2}$ and $b(f)=\frac{\langle\mathcal{B}_L f,f\rangle}{\|f\|_2^2}$. Since $\mathcal{B}_L$ is self-adjoint, $b(f)$ is real. Due to Theorem 3.1 in [11] the equality for the uncertainty principle is attained if and only if there exist $\lambda\in\mathbb{C}$ such that

$$(\mathcal{B}_L-b(f))f=\lambda(\mathcal{A}_L-a(f))f=-\overline{\lambda}(\mathcal{A}_L^*-\overline{a(f)})f.$$

The second identity yields

$$f(x)\left(\lambda\mathrm{e}^{2\pi\mathrm{i}\langle L,x\rangle}+\overline{\lambda}\mathrm{e}^{-2\pi\mathrm{i}\langle L,x\rangle}-a(f)\lambda-\overline{\lambda a(f)}\right)$$
$$=2f(x)(\mathcal{R}e(\lambda\mathrm{e}^{2\pi\mathrm{i}\langle L,x\rangle})-\mathcal{R}e(a(f)\lambda))\equiv 0.$$

This condition can be satisfied only if $f=0$ or $\lambda=0$. For the second case, we get $(\mathcal{B}_L-b(f))f=0$ or $\frac{\mathrm{i}}{2\pi}\frac{\partial f}{\partial L}(x)=b(f)f(x)$. If $\frac{\partial f}{\partial L}(x)\neq 0$, then comparing Fourier coefficients we conclude that f is a monomial, i.e. $f(x)=C\mathrm{e}^{2\pi\mathrm{i}\langle k,\xi\rangle}$. Recall that for monomials the directional UP is not defined. If $\frac{\partial f}{\partial L}(x)=0$, then $b(f)=0$, and the equation $(\mathcal{B}_L-b(f))f=0$ holds. The general solution of the equation $\frac{\partial f}{\partial L}(x)=0$ is the function $f(x)=\Phi(L_2x_1-L_1x_2, L_3x_2-L_2x_3,\dots,L_dx_{d-1}-L_{d-1}x_d)$, where $\Phi(x)$ is a differentiable function.

Let us compare the UP defined by Goh and Goodman and the directional UP. They are not equivalent. The next lemma gives a pair of examples where the UP's behave differently.

Lemma 1: Let $L\in\mathbb{Z}^d$.

1) Suppose $\widetilde{p}_n(x)=(1+\cos 2\pi\langle L,x\rangle)^n+2\cos 2\pi x_1$, where $|L_j|>1$ for all $j=1,...,d$, and if $d=1$, then L is not collinear to e_1. Then

$$\mathrm{UP}_L^{\mathbb{T}^d}(\widetilde{p}_n)\to\frac{1}{4},\qquad\frac{\mathrm{UP}_{GG}^{\mathbb{T}^d}(\widetilde{p}_n)}{n\ 4^n}\to\frac{d\|L\|^2}{32}\quad n\to\infty.$$

2) Suppose $\widetilde{t}_n(x)=(1+\cos 2\pi x_1)^n+2\cos 2\pi\langle L,x\rangle$, where $|L_j|>1$ for all $j=1,...,d$, and if $d=1$, then L is not collinear to e_1. Then

$$\frac{\mathrm{UP}_L^{\mathbb{T}^d}(\widetilde{t}_n)}{n\ 4^n}\to\frac{L_1^2}{32\|L\|^4},\quad\frac{\mathrm{UP}_{GG}^{\mathbb{T}^d}(\widetilde{t}_n)}{n}\to\frac{d-1}{4}\ \ n\to\infty.$$

Proof. Let us prove item 1). For convenience, we will use the notation $p_n(x)=(1+\cos 2\pi\langle L,x\rangle)^n$. Denote $I_n:=\int_{\mathbb{T}^d}p_n(x)\mathrm{d}x$. Since $p_n(x)=2^n\cos^{2n}(\pi\langle L,x\rangle)=2^{-n}\left(e^{\pi i\langle L,x\rangle}+e^{-\pi i\langle L,x\rangle}\right)^{2n}$ it follows that $I_n=\frac{(2n-1)!!}{n!}$. Then

$$\|\widetilde{p}_n\|_{\mathbb{T}^d}^2=\|p_n\|_{\mathbb{T}^d}^2+2=\frac{(4n-1)!!}{(2n)!}+2,$$

$$\langle\mathcal{A}_L\widetilde{p}_n,\widetilde{p}_n\rangle=\langle\mathcal{A}_Lp_n,p_n\rangle=I_{2n+1}-I_{2n},$$

$$\mathcal{B}_L\widetilde{p}_n(x)=-i\|L\|^2n(1+\cos(2\pi\langle L,x\rangle))^{n-1}\sin(2\pi\langle L,x\rangle)$$
$$-2iL_1\sin 2\pi x_1,$$

$$\|\mathcal{B}_L\widetilde{p}_n\|_{\mathbb{T}^d}^2=n^2\|L\|^4(2I_{2n-1}-I_{2n})+2L_1^2.$$

Since $\widetilde{p}_n$ is even and $\mathcal{B}_L\widetilde{p}_n$ is odd we get $\langle\mathcal{B}_L\widetilde{p}_n,\widetilde{p}_n\rangle=0$. Therefore,

$$\mathrm{UP}_L^{\mathbb{T}^d}(\widetilde{p}_n)=\frac{1}{\|L\|^4}\left(\frac{\left(\frac{(4n-1)!!}{(2n)!}+2\right)^2}{\left(2n\frac{(4n-1)!!}{(2n+1)!}\right)^2}-1\right)\left(\frac{n^2\|L\|^4\frac{(4n-3)!!}{(2n)!}+2L_1^2}{\frac{(4n-1)!!}{(2n)!}+2}\right)$$

$$=\frac{n^2}{(2n+1)(4n-1)}\frac{\left(1+2\frac{(2n)!(2n+1)}{(4n-1)!!}\right)\left(2+2\frac{(2n)!}{(4n-1)!!}-\frac{1}{2n+1}\right)}{\left(\frac{2n}{2n+1}\right)^2}$$

$$\left(\frac{1+2\frac{L_1^2}{\|L\|^4}\frac{(2n)!(4n-1)}{n^2(4n-1)!!}}{1+2\frac{(2n)!}{(4n-1)!!}}\right).$$

By the Stirling formula $n!=\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n(1+O(1/n))$, it follows that $\frac{(2n)!(2n+1)}{(4n-1)!!}=\frac{2n\sqrt{2\pi n}(1+O(\frac{1}{n}))}{2^{2n}}\to 0,\ n\to\infty$. Therefore, $\mathrm{UP}_L^{\mathbb{T}^d}(\widetilde{p}_n)\to\frac{1}{4},\quad n\to\infty$.

Now, we compute $\mathrm{UP}_{GG}^{\mathbb{T}^d}(\widetilde{p}_n)$. Let $\widetilde{c}_k=\widetilde{c}_k(\widetilde{p}_n)$ be the Fourier coefficients of $\widetilde{p}_n$. Then

$$\widetilde{c}_0=\int_{\mathbb{T}^d}\widetilde{p}_n(x)\mathrm{d}x=\int_{\mathbb{T}^d}p_n(x)\mathrm{d}x=I_n=\frac{(2n-1)!!}{n!},$$

$$\langle\mathcal{A}_j\widetilde{p}_n,\widetilde{p}_n\rangle=\sum_{k\in\mathbb{Z}^d}\widetilde{c}_{k-e_j}\widetilde{c}_k=\delta_{j,1}(\widetilde{c}_{-e_1}\widetilde{c}_{\mathbf{0}}+\widetilde{c}_{\mathbf{0}}\widetilde{c}_{e_1})$$

$$=2\delta_{j,1}\frac{(2n-1)!!}{n!},\ \text{for } j=1,\dots,d.$$

Further,

$$\mathcal{B}_j\widetilde{p}_n(x)=-\mathrm{i}L_jn(1+\cos(2\pi\langle L,x\rangle))^{n-1}\sin(2\pi\langle L,x\rangle)$$

$$-2\mathrm{i}\delta_{j,1}\sin 2\pi x_1.$$

Therefore, $\|\mathcal{B}_j\widetilde{p}_n\|^2_{\mathbb{T}^d} = n^2L_j^2(2I_{2n-1}-I_{2n})+2\delta_{j,1}$. Since $\widetilde{p}_n$ is even and $\mathcal{B}_j\widetilde{p}_n$ is odd, we get $\langle\mathcal{B}_j\widetilde{p}_n,\widetilde{p}_n\rangle = 0$. Hence, combining all results in the definition of $\mathrm{UP}_{GG}^{\mathbb{T}^d}(\widetilde{p}_n)$ (2) and after some simplifications, we obtain

$$\mathrm{UP}_{GG}^{\mathbb{T}^d}(\widetilde{p}_n) = \frac{n^2\|L\|^2}{4(4n-1)}$$

$$\left(d\left(\frac{(4n-1)!!}{(2n)!}\frac{n!}{(2n-1)!!}+2\frac{n!}{(2n-1)!!}\right)^2-4\right)\frac{1+\frac{2(2n)!}{n^2\|L\|^2(4n-1)!!}}{1+2\frac{(2n)!}{(4n-1)!!}}.$$

By the Stirling formula $\frac{(2n)!}{(4n-1)!!} = \frac{\sqrt{2\pi n}(1+O(\frac{1}{n}))}{2^{2n}} \to 0$ as $n\to\infty$ and $\frac{n!}{(2n-1)!!} = \frac{\sqrt{\pi n}(1+O(\frac{1}{n}))}{2^n} \to 0$ as $n\to\infty$. Thus, $\frac{(4n-1)!!}{(2n)!}\frac{n!}{(2n-1)!!} = \frac{2^n}{\sqrt{2}}(1+O(\frac{1}{n}))$ as $n\to\infty$. Finally, it follows that $\frac{\mathrm{UP}_{GG}^{\mathbb{T}^d}(\widetilde{p}_n)}{n4^n} \to \frac{d\|L\|^2}{32}$ as $n\to\infty$.

Item 2) can be proved analogously. By similar arguments it can be shown that

$$\mathrm{UP}_{L}^{\mathbb{T}^d}(\widetilde{t}_n) = \frac{1}{\|L\|^4}\left(\left(\frac{\frac{(4n-1)!!}{(2n)!}}{\frac{(2n-1)!!}{n!}}\right)^2\frac{\left(1+2\frac{(2n)!}{(4n-1)!!}\right)^2}{4}-1\right)$$

$$\frac{L_1^2/2+2\|L\|^4\frac{(2n-1)!}{n(4n-3)!!}}{1+2\frac{(2n)!}{(4n-1)!!}}\frac{2n^2}{4n-1}$$

and

$$\mathrm{UP}_{GG}^{\mathbb{T}^d}(\widetilde{t}_n) = \left(d\left(\frac{2n+1}{2n}+2\frac{(2n)!}{(4n-1)!!}\frac{2n+1}{2n}\right)^2-1\right)$$

$$\frac{\frac{n}{2}+2\|L\|^2\frac{(2n-1)!}{(4n-3)!!}}{1+2\frac{(2n)!}{(4n-1)!!}}\frac{2n}{4n-1}.$$

The Stirling formula yields Item 2).$\Diamond$

Consider several cases which are excluded in Lemma 1 and the behavior of UP's. If in item 1) L is collinear to e_1, then L is not collinear to e_2 and functionals $\mathrm{UP}_{L}^{\mathbb{T}^d}$ and $\mathrm{UP}_{GG}^{\mathbb{T}^d}$ for a sequence of trigonometric polynomials $(1+\cos 2\pi\langle L,x\rangle)^n + 2\cos 2\pi x_2$, $n\in\mathbb{N}$, have the same behavior as in item 1). Next, suppose L is not collinear to e_1, but $L_i = 1$ for some i. It is more hard to deal with all cases analytically, but for $d=2$ the behavior can be caught numerically. If $L=(1,1)$, then the behavior is the same, but $\frac{\mathrm{UP}_{GG}^{\mathbb{T}^d}(\widetilde{p}_n)}{n\,4^n}$ tends to a different constant. If $L=(0,1)$, then $\mathrm{UP}_{L}^{\mathbb{T}^d}(\widetilde{p}_n)$ again tends to $1/4$ and $\mathrm{UP}_{GG}^{\mathbb{T}^d}(\widetilde{p}_n)$ grows linearly (not exponentially as in item 1)). Two latter cases are plotted in Fig. 1, 1st row.

Now, consider item 2). When $L=(1,1)$, numerically the behavior is the same as in Lemma 1. However, if $L=(1,0)$, then $\mathrm{UP}_{L}^{\mathbb{T}^d}(\widetilde{t}_n)$ tends to $1/4$ and $\mathrm{UP}_{GG}^{\mathbb{T}^d}(\widetilde{t}_n)$ has the same behavior as in item 2). If $L=(0,1)$ both UP's grow linearly. Two latter cases are plotted in Fig. 1, 2nd row.

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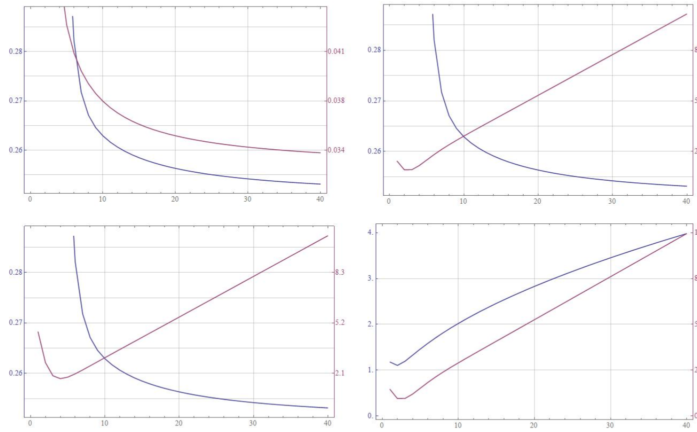

Fig. 1. Horizontal axis indicates the order n of a polynomial. Vertical axis indicates values of UP's. Left scale on a vertical axis and blue color is for $\mathrm{UP}_{L}^{\mathbb{T}^d}$-case, right scale on a vertical axis and purple color is for $\mathrm{UP}_{GG}^{\mathbb{T}^d}$-case, $d=2$. Top, left: $L=(1,1)$, $\mathrm{UP}_{L}^{\mathbb{T}^d}(\tilde{p}_n)$, $\mathrm{UP}_{GG}^{\mathbb{T}^d}(\tilde{p}_n)/(n\cdot 4^n)$. Top, right: $L=(0,1)$, $\mathrm{UP}_{L}^{\mathbb{T}^d}(\tilde{p}_n)$, $\mathrm{UP}_{GG}^{\mathbb{T}^d}(\tilde{p}_n)$. Bottom, left: $L=(1,0)$, $\mathrm{UP}_{L}^{\mathbb{T}^d}(\tilde{t}_n)$, $\mathrm{UP}_{GG}^{\mathbb{T}^d}(\tilde{t}_n)$. Bottom, right: $L=(0,1)$, $\mathrm{UP}_{L}^{\mathbb{T}^d}(\tilde{t}_n)$, $\mathrm{UP}_{GG}^{\mathbb{T}^d}(\tilde{t}_n)$.

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