A directional periodic uncertainty principle

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Abstract—A notion of a directional uncertainty product (UP) for multivariate periodic functions is introduced. It is a characteristic of a localization for a signal along a fixed direction.

I. INTRODUCTION

A notion of uncertainty product (UP) is a sufficiently well-studied object in harmonic analysis. Initially, it was introduced for functions on the real line to measure a simultaneous localization of a function and its Fourier transform [1]. The essence of this quantification of localization is contained in the Heisenberg uncertainty principle, which says that for any appropriate function the UP cannot be smaller than a positive absolute constant. Later, numerous versions of this general principle were developed for different algebraic and topological structures such as abstract locally compact groups, high-dimensional spheres, etc. (see, e.g., [2], [3], [4]). For more detailed information concerning this topic, we refer the interested reader to surveys [5] and [6] and the references therein.

In this paper we focus on the case of multivariate periodic functions and multivariate discrete signals. For periodic functions of one variable a notion of UP was introduced in 1985 by Breitenberger in [7]. The corresponding uncertainty principle is also valid in this setup. One possible extension of this notion to the case of multivariate periodic functions was suggested by Goh and Goodman in [8] (see formula (2)). However, this approach does not take into account the main difference between periodic functions of one variable and many variables, namely the localization of a function along particular directions. The main contribution of this paper is a new approach that allows to include the directionality into the definition of the UP (see formula (3)). We compare these two approaches and discuss the differences in detail (see also [9]). At the same time, both definitions fit into a more general operator approach (see formula (1)). This approach was established by Folland in [10] and was extended to two normal or symmetric operators by Selig in [11] and Goh, Micchelli in [12]. For several operators this approach was generalized by Goh and Goodman in [8].

II. BASIC NOTATIONS AND DEFINITIONS

We use the standard multi-index notation. Let \( d \in \mathbb{N} \), \( \mathbb{R}^d \) be the \( d \)-dimensional Euclidean space, \( \{ e_j, 1 \leq j \leq d \} \) be the standard basis in \( \mathbb{R}^d \), \( \mathbb{Z}^d \) be the integer lattice in \( \mathbb{R}^d \), \( \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d \) be the \( d \)-dimensional torus. Let \( x = (x_1, \ldots, x_d)^T \) and \( y = (y_1, \ldots, y_d)^T \) be column vectors in \( \mathbb{R}^d \). Then \( \langle x, y \rangle := x_1y_1 + \cdots + x_dy_d \), \( \| x \| := \sqrt{\langle x, x \rangle} \), \( \| x \|_1 = \sum_{j=1}^d |x_j| \), \( \| x \|_\infty = \max_j |x_j| \). We say that \( x \geq y \), if \( x_j \geq y_j \) for all \( j = 1, \ldots, d \), and we say that \( x > y \), if \( x \geq y \) and \( x \neq y \). Further, \( \mathbb{Z}^d_+ := \{ \alpha \in \mathbb{Z}^d : \alpha \geq 0 \} \), where \( 0 = (0, \ldots, 0) \) denotes the origin in \( \mathbb{R}^d \). For \( \alpha = (\alpha_1, \ldots, \alpha_d)^T \in \mathbb{Z}^d_+ \), denote \( |\alpha| := \alpha_1 + \cdots + \alpha_d \).

For a sufficiently smooth function \( f \) defined on \( \Omega \subset \mathbb{R}^d \) and a multi-index \( \alpha \in \mathbb{Z}^d_+ \), \( D^\alpha f \) denotes the derivative of \( f \) of order \( \alpha \) and \( D^\alpha f = \frac{\partial^{\alpha}}{\partial x^\alpha} f = \frac{\partial^{\alpha}}{\partial y^\alpha} f \). For \( \alpha = e_j \), we also use \( D^\alpha f = f_j \). The directional derivative of a sufficiently smooth function \( f \) defined on \( \Omega \) along a vector \( L = (L_1, \ldots, L_d) \in \mathbb{R}^d \) is denoted by \( \frac{Df}{D_L} = \sum_{j=1}^d L_j \frac{Df}{Dx_j} \).

For a function \( f \in L_2(\mathbb{T}^d) \) its norm is denoted by \( \| f \|_{L_2}^2 := \int_{\mathbb{T}^d} |f(x)|^2 \, dx \). The Fourier series coefficients of a function \( f \in L_2(\mathbb{T}^d) \) are given by \( c_k = c_k(f) = \hat{f}(k) = \int_{\mathbb{T}^d} f(x)e^{-2\pi i k \cdot x} \, dx, \ k \in \mathbb{Z}^d \). The Sobolev space \( H^1(\mathbb{T}^d) \) consists of functions in \( L_2(\mathbb{T}^d) \) such that all its derivatives of the first order are also in \( L_2(\mathbb{T}^d) \), which can be written as

\[
H^1(\mathbb{T}^d) = \left\{ f \in L_2(\mathbb{T}^d) : \sum_{k \in \mathbb{Z}^d} \| k \|^2 |c_k(f)|^2 < \infty \right\}
\]

Let \( \mathcal{H} \) be a Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and with norm \( \| \cdot \| := \langle \cdot, \cdot \rangle^{1/2} \). Let \( A, B \) be two linear operators with domains \( D(A), D(B) \subseteq \mathcal{H} \) and ranges in \( \mathcal{H} \). The variance of non-zero \( f \in D(A) \) with respect to the operator \( A \) is defined to be

\[
\Delta(A, f) = \| A f \|^2 - \| (Af, f) \|_2^2 = \left\| \left( A - \frac{(Af, f)}{\| f \|^2} \right) f \right\|^2 = \min_{\alpha \in \mathbb{C}} \| Af - \alpha f \|^2.
\]

We recall that a densely defined linear operator \( A \) in a Hilbert space \( \mathcal{H} \) is said to be symmetric if \( \langle Af, g \rangle = \langle f, Ag \rangle \) for \( f, g \in D(A) \). If additionally \( D(A) = D(A^*) \), where \( A^* \) is an adjoint operator for \( A \), then \( A \) is self-adjoint. We say that \( A \) is normal if \( A \) is closed, densely defined and if \( A^* A = A A^* \). For a normal operator \( A \) we have that \( D(A) = D(A^*) \) and \( \| Af \| = \| A^* f \| \) for any \( f \in D(A) \).

The commutator of \( A \) and \( B \) is defined by \( [A, B] := AB - BA \) with domain \( D(AB) \cap D(BA) \).
**Theorem 1:** [8, Theorem 4.1] Let $A_1, \ldots, A_n, B_1, \ldots, B_n$ be symmetric or normal operators acting from a Hilbert space $\mathcal{H}$ into itself. Then for any non-zero $f$ in $\mathcal{D}(A_jB_j) \cap \mathcal{D}(B_jA_j)$, $j = 1, \ldots, n$,

\[
\frac{1}{4} \left( \sum_{j=1}^{n} |\langle [A_j, B_j]f, f \rangle| \right) \leq \sum_{j=1}^{n} \Delta(A_j, f) \sum_{j=1}^{n} \Delta(B_j, f). \tag{1}
\]

We recall that for two operators $A, B$ on a Hilbert space $\mathcal{H}$, which are symmetric or normal, the uncertainty principle was established by Selig in [11, Theorem 3.1] as

\[
\frac{1}{4} |\langle [A, B]f, f \rangle|^2 \leq \Delta_A(f) \Delta_B(f)
\]

for all non-zero $f \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$.

If the commutator $\langle [A_j, B_j]f, f \rangle$ is non-zero for all $j = 1, \ldots, n$, then the UP for $f$ is defined as

\[
\text{UP}(f) := \frac{\left( \sum_{j=1}^{n} \Delta(A_j, f) \right) \left( \sum_{j=1}^{n} \Delta(B_j, f) \right)}{\left( \sum_{j=1}^{n} |\langle [A_j, B_j]f, f \rangle| \right)^2}.
\]

In this terms, the uncertainty principle says that $\text{UP}(f)$ cannot be smaller than $\frac{1}{4}$, for any appropriate function $f$.

The well-known Heisenberg UP for functions defined on the real line fits in this operator approach, if $n = 1$, $\mathcal{H} = L_2(\mathbb{R})$ and the two operators are as follows $Af(x) = 2\pi xf(x)$, $Bf(x) = \frac{1}{2\pi} \frac{df}{dx}(x)$.

The Breitenberger UP is defined for periodic functions. In this case, $n = 1$, $\mathcal{H} = L_2(\mathbb{T})$ and $\mathcal{A}^T f(x) = e^{2\pi i x} f(x)$, $\mathcal{B}^T f(x) = \frac{1}{2\pi} \frac{df}{dx}(x)$. The commutator is $[\mathcal{A}^T, \mathcal{B}^T] = \mathcal{A}^T$. It is more convenient for the Breitenberger UP to use the notions of the angular and frequency variance. Since $\|\mathcal{A}^T f\|_2^2 = \|f\|_2^2$,

\[
\text{var}^A(f) := \frac{\|f\|_2^2 \Delta(\mathcal{A}^T, f)}{\|\mathcal{A}^T, \mathcal{B}^T f\|_2^2} = \left( \frac{\|f\|_2^2}{\|\mathcal{A}^T f\|_2^2} \right)^2 - 1
\]

\[
\text{var}^F(f) := \frac{\|\mathcal{B}^T f\|_2^2}{\|f\|_2^2} = \frac{\|\mathcal{B}^T f\|_2^2}{\|f\|_2^2} - \frac{\|\mathcal{B}^T f\|_2^2}{\|f\|_2^2}
\]

\[
\text{UP}^\mathcal{T}(f) := \text{var}^A(f) \text{var}^F(f).
\]

It is known that the lower bound for $\text{UP}^\mathcal{T}$ is not attained at any function. But there exist sequences of functions such that $\text{UP}^\mathcal{T}$ tends to the optimal value $\frac{1}{4}$ (see, e.g., [13]).

**Theorem 2:** For a function $f \in H^1(\mathbb{T}^d)$, such that $\langle A_j f, f \rangle \neq 0$ for some $j = 1, \ldots, d$, the functional $\text{UP}^\mathcal{T}_{GCG}(f)$ is well-defined and

\[
\text{UP}^\mathcal{T}_{GCG}(f) = \frac{1}{\|f\|_2^4} \sum_{j=1}^{d} \left( \frac{\|f\|_2^4}{\|\mathcal{A}_j f\|_2^2} - \frac{\|\mathcal{B}_j f\|_2^2}{\|f\|_2^4} \right) - \frac{1}{2} \sum_{j=1}^{d} \frac{\|f\|_2^4}{\|\mathcal{B}_j f\|_2^2} + \frac{1}{2} \sum_{j=1}^{d} \frac{\|\mathcal{A}_j f\|_2^2}{\|f\|_2^4}.
\]

It can be shown, that the variances attain the value $\infty$ if and only if $\langle A_j f, f \rangle = 0$, for all $j = 1, \ldots, d$. In these cases, we can also assign to $\text{UP}^\mathcal{T}_{GCG}(f)$ the value $\infty$, except the following case var$^G_{GCG}(f) = 0$ and var$^A_{GCG}(f) = \infty$. This case happens if and only if $f$ is a monomial. Indeed, var$^G_{GCG}(f) = 0$ if and only if $\Delta(\mathcal{B}_j f) = 0$ for all $j = 0, \ldots, d$, that implies $\mathcal{B}_j f = \beta_j f$ for some $\beta_j \in \mathbb{C}$, that is $\frac{df}{dx} = \alpha_j f$ for some $\alpha_j \in \mathbb{C}$. Since $f \in H^1(\mathbb{T}^d)$, it follows that $f$ is a monomial. However, in this case, i.e., var$^G_{GCG}(f) = 0$ and var$^A_{GCG}(f) = \infty$, inequality (1) takes the form $1/4 \cdot 0 \leq C \cdot 0$. It is trivially true. Thus, inequality (1) is valid for all non-zero functions $f \in H^1(\mathbb{T}^d)$.

In fact, the above approach for the definition of the UP does not deal with a new phenomenon, that appears in the multidimensional case, namely, the localization of a function along particular directions. We suggest an approach that allows to include the directionality into the definition.

The directional UP for $\mathcal{T}^d$ along a direction $L \in \mathbb{Z}^d$ ($L \neq 0$) is defined using the operators

\[
\mathcal{A}_L f(x) = e^{2\pi i L \cdot x} f(x), \quad \mathcal{B}_L f(x) = \frac{\partial}{\partial L} f(x).
\]

with domains $\mathcal{D}(\mathcal{A}_L) = L_2(\mathbb{T}^d)$, $\mathcal{D}(\mathcal{B}_L) = H^1(\mathbb{T}^d)$. Note that $\mathcal{A}_L$ is normal, $\mathcal{B}_L$ is symmetric. The commutator for $f \in \mathcal{D}(\mathcal{A}_L) \cap \mathcal{D}(\mathcal{B}_L)$ is $[\mathcal{A}_L, \mathcal{B}_L] f = \|L\|^2 \mathcal{A}_L f$. Thus, the directional UP for a function $f \in \mathcal{D}(\mathcal{A}_L) \cap \mathcal{D}(\mathcal{B}_L)$ such that $\mathcal{A}_L f \neq 0$ is defined as

\[
\text{UP}^\mathcal{T}_{L}(f) = \frac{1}{\|L\|^2} \left( \frac{\|f\|_2^4}{\|\mathcal{A}_L f\|_2^2} - 1 \right) \left( \frac{\|\mathcal{B}_L f\|_2^2}{\|f\|_2^4} - \frac{\|\mathcal{B}_L f\|_2^2}{\|f\|_2^4} \right) := \frac{1}{\|L\|^2} \text{var}^A(f) \text{var}^F(f),
\]

III. MAIN RESULTS

For the space $L_2(\mathbb{T}^d)$ of multivariate periodic functions, Goh and Goodman in [8] suggest to take the operators as follows $A_j f(x) = e^{2\pi i j_x} f(x)$, $B_j f(x) = \frac{1}{2\pi} \frac{df}{dx} (x), j = 1, \ldots, d$. Note that the domains of the operators are $\mathcal{D}(A_j) = L_2(\mathbb{T}^d)$, $\mathcal{D}(B_j) = H^1(\mathbb{T}^d)$. Operators $A_j$ are normal, $B_j$ are self-adjoint. The commutators for $f \in H^1(\mathbb{T}^d)$ are $[A_j, B_j] f = A_j f$. The uncertainty principle for these operators is stated as follows.
where $\var^L_1(f)$ is the angular directional variance and $\var^L_2(f)$ is the frequency directional variance.

Theorem 3: For $L \in \mathbb{Z}^d$ and a function $f \in H^1(T^d)$, such that $(A_L f, f) \neq 0$, the functional $\UP_L^{\var^L_2}(f)$ is well-defined and

$$
\UP_L^{\var^L_2}(f) = \frac{1}{||L||^4} \left( \frac{\sum_{k \in \mathbb{Z}^d} |c_k|^2}{\sum_{k \in \mathbb{Z}^d} c_k \cdot L_k e^{2\pi i(k,x)}} - 1 \right)
$$

where $c_k = c(k)$ are the Fourier coefficients of $f$.

The statement easily follows from the operator approach and

$$
A_L f(x) = \sum_{k \in \mathbb{Z}^d} c_{k-L} e^{2\pi i(k,x)} ,
$$

$$
B_L f(x) = -\sum_{k \in \mathbb{Z}^d} (L,k) c_k e^{2\pi i(k,x)} .
$$

It can be shown, that the directional variances attain the value $\infty$ if and only if $(A_L f, f) = 0$. In this case, we can also assign to $\UP_L^{\var^L_2}(f)$ the value $\infty$, except the following cases $\var^L_2(f) = 0$ and $\var^L_1(f) = \infty$. However in this case, (1) is trivially satisfied $(0 \leq 0)$, so inequality (1) is valid for operators $A_L$ and $B_L$ for all non-zero functions $f \in H^1(T^d)$.

In contrast to the Breitenberger UP and to the UP defined by Goh and Goodman, the optimal function exists for the directional UP. Indeed, let $a(f) = \frac{\langle A_L f, f \rangle}{\|f\|^2_L}$ and $b(f) = \frac{\langle B_L f, f \rangle}{\|f\|^2_L}$. Since $B_L$ is self-adjoint, $b(f)$ is real. Due to Theorem 3.1 in [11] the equality for the uncertainty principle is attained if and only if exist $\lambda \in \mathbb{C}$ such that

$$(B_L - b(f)) f = \lambda (A_L - a(f)) f = -\lambda (A_L^* - a(f)) f .$$

The second identity yields

$$
f(x) \left( e^{2\pi i(L,x)} + \lambda e^{-2\pi i(L,x)} - a(f) \lambda - \lambda a(f) \right) = 2f(x)(\Re(e^{2\pi i(L,x)}) - \Re(a(f) \lambda)) \equiv 0 .
$$

This condition can be satisfied only if $f = 0$ or $\lambda = 0$. For the second case, we get $(B_L - b(f)) f = 0$ or $\frac{\partial f}{\partial x} (x) \neq 0$, then comparing Fourier coefficients we conclude that $f$ is a monomial, i.e. $f(x) = C e^{2\pi i(k,x)}$. Recalling that for monomials the directional UP is not defined. If $\frac{\partial f}{\partial x} (x) = 0$, then $b(f) = 0$, and the equation $\langle B_L b(f) \rangle f = 0$ holds. The general solution of the equation $\frac{\partial f}{\partial x} (x) = 0$ is the function $f(x) = \Phi(L_2 x_1 - L_1 x_2, L_3 x_2 - L_2 x_3, \ldots, L_d x_{d-1} - L_{d-1} x_d)$, where $\Phi(x)$ is a differentiable function.

Let us compare the UP defined by Goh and Goodman and the directional UP. They are not equivalent. The next lemma gives a pair of examples where the UP’s behave differently.

Lemma 1: Let $L \in \mathbb{Z}^d$.

1) Suppose $\tilde{p}_n(x) = (1 + \cos 2\pi \langle L, x \rangle)^n + 2 \cos 2\pi x_1$, where $|L_j| > 1$ for all $j = 1, \ldots, d$, and if $d = 1$, then $L$ is not collinear to $e_1$. Then

$$
\UP_L^{\var^L_2}(\tilde{p}_n) \to \frac{1}{4}, \quad \UP_G^{\var^L_2}(\tilde{p}_n) \to \frac{d||L||^2}{32} n \to \infty .
$$

2) Suppose $\tilde{p}_n(x) = (1 + \cos 2\pi x_1)^n + 2 \cos 2\pi \langle L, x \rangle$, where $|L_j| > 1$ for all $j = 1, \ldots, d$, and if $d = 1$, then $L$ is not collinear to $e_1$. Then

$$
\UP_L^{\var^L_2}(\tilde{p}_n) \to \frac{L_1^2}{32 d ||L||^4} , \quad \UP_G^{\var^L_2}(\tilde{p}_n) \to \frac{d - 1}{4} n \to \infty .
$$

Proof. Let us prove item 1). For convenience, we will use the notation $p_n(x) = (1 + \cos 2\pi \langle L, x \rangle)^n$. Denote $I_n := \int_{T^d} p_n(x) dx$. Since $p_n(x) = 2^n \cos^{2n}(\pi \langle L, x \rangle) = 2^{-n} (e^{\pi \langle L, x \rangle} + e^{-\pi \langle L, x \rangle})^{2n}$, it follows that $I_n = \frac{(2n - 1)!!}{n!}$. Then

$$
||\tilde{p}_n||^2_{T^d} = \|p_n||^2_{T^d} + 2 = \frac{(4n - 1)!!}{(2n)!!} + 2 , \quad \langle A_L \tilde{p}_n, \tilde{p}_n \rangle = \langle A_L p_n, p_n \rangle = I_{2n+1} - I_{2n} .
$$

$$
B_L \tilde{p}_n(x) = -i \langle L ||L||^2 n(1 + \cos(2\pi \langle L, x \rangle))^n - 1) \sin(2\pi \langle L, x \rangle)
$$

$$
-2li \sin 2\pi x_1 , \quad ||B_L \tilde{p}_n||^2_{T^d} = \|p_n||^2_{T^d} + 2L_1^2 .
$$

Since $\tilde{p}_n$ is even and $B_L \tilde{p}_n$ is odd we get $\langle B_L \tilde{p}_n, \tilde{p}_n \rangle = 0$. Therefore,

$$
\UP_L^{\var^L_2}(\tilde{p}_n) = \frac{1}{||L||^4} \left( \frac{\langle \tilde{p}_n ||L||^2 (4n - 1)!! + 2 \cos 2\pi \langle L, x \rangle \rangle}{\|p_n||^2_{T^d} + 2L_1^2} \right)
$$

$$
= \frac{2n}{(2n + 1)(4n - 1)} \left( \frac{\langle \tilde{p}_n ||L||^2 (4n - 1)!! + 2 \cos 2\pi \langle L, x \rangle \rangle}{\|p_n||^2_{T^d} + 2L_1^2} \right)
$$

$$
= \frac{2n}{(2n + 1)(4n - 1)} \left( \frac{\langle \tilde{p}_n ||L||^2 (4n - 1)!! + 2 \cos 2\pi \langle L, x \rangle \rangle}{\|p_n||^2_{T^d} + 2L_1^2} \right)
$$

By the Stirling formula $n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n (1 + O(1/n))$, it follows that $\langle \tilde{p}_n ||L||^2 (4n - 1)!! + 2 \cos 2\pi \langle L, x \rangle \rangle \to 0$, $n \to \infty$. Therefore,

$$
\UP_G^{\var^L_2}(\tilde{p}_n) \to \frac{1}{4} n \to \infty .
$$

Now, we compute $\UP_G^{\var^L_2}(\tilde{p}_n)$. Let $\tilde{c}_k = \tilde{c}^*_k(p_n)$ be the Fourier coefficients of $\tilde{p}_n$. Then

$$
\tilde{c}_0 = \int_{T^d} \tilde{p}_n(x) dx = \int_{T^d} p_n(x) dx = I_n = \frac{(2n - 1)!!}{n!} ,
$$

$$
\langle A_L \tilde{p}_n, \tilde{p}_n \rangle = \sum_{k \in \mathbb{Z}^d} \tilde{c}_{k-L} \tilde{c}_k = \delta_{j,1} (\tilde{c}_{e_1} \tilde{c}_0 + \tilde{c}_0 \tilde{c}_{e_1})
$$

$$
= 2\delta_{j,1} \left( \frac{(2n - 1)!!}{n!} \right) , \text{ for } j = 1, \ldots, d .
$$

Further,

$$
B_j \tilde{p}_n(x) = -i L_j n(1 + \cos(2\pi \langle L, x \rangle))^n - 1) \sin(2\pi \langle L, x \rangle)
$$
Therefore, $\| B_j \tilde{p}_n \|^2_{L^d} = n^2 L_j^2 (2I_{2n-1} - I_{2n}) + 2d \delta_{j,1}$. Since $\tilde{p}_n$ is even and $B_j \tilde{p}_n$ is odd, we get $\langle B_j \tilde{p}_n, \tilde{p}_n \rangle = 0$. Hence, combining all results in the definition of $\text{UP}_{G_G}^d(\tilde{p}_n)$ (2) and after some simplifications, we obtain

$$\text{UP}_{G_G}^d(\tilde{p}_n) = \frac{n^2 \| L \|^2_{L^d}}{4(4n-1)} \left( d \left( \frac{(4n-1)!}{(2n)!} \frac{n!}{(2n-1)!} + 2 \frac{n!}{(2n-1)!} \right)^2 - 4 \right) \frac{1 + \frac{2(2n)!}{n^2 \| L \|^2_{L^d}(4n-1)!}}{1 + 2 \frac{2(2n)!}{(4n-1)!}}.$$

By the Stirling formula $\frac{2n!}{(2n-1)!} = \sqrt{2\pi n} (1 + O(\frac{1}{n})) \rightarrow 0$ as $n \rightarrow \infty$ and $\frac{n!}{(2n-1)!} = \frac{2\sqrt{\pi n}}{2^n} (1 + O(\frac{1}{n})) \rightarrow 0$ as $n \rightarrow \infty$. Thus, it follows that $\text{UP}_{G_G}^d(\tilde{p}_n) \rightarrow d \| L \|^2_{L^d}$ as $n \rightarrow \infty$.

Item 2) can be proved analogously. By similar arguments it can be shown that

$$\text{UP}_{L}^d(\tilde{t}_n) = \frac{1}{\| L \|^2_{L^d}} \left( \frac{(4n-1)!!}{(2n-1)!!} \frac{2n}{n!} \left( 1 + \frac{2(2n)!}{n^2 \| L \|^2_{L^d}(4n-1)!} \right)^2 - 1 \right) \left( \frac{4}{4n-1} \right)$$

and

$$\text{UP}_{G_G}^d(\tilde{t}_n) = \left( d \left( \frac{2n + 1}{2n} + 2 \frac{(2n)!}{(4n-1)!} \frac{2n}{n!} \right)^2 - 1 \right) \left( \frac{2n}{n!} \right)^2 \left( \frac{4}{4n-1} \right)$$

The Stirling formula yields Item 2).\( \diamond \)

Consider several cases which are excluded in Lemma 1 and the behavior of UP's. If in item 1) $L$ is collinear to $e_1$, then $L$ is not collinear to $e_2$ and functionals $\text{UP}_{L}^d$ and $\text{UP}_{G_G}^d$ for a sequence of trigonometric polynomials $1 + \cos 2\pi x, \sin 2\pi x, n \in \mathbb{N}$, have the same behavior as in item 1). Next, suppose $L$ is not collinear to $e_1$, but $L_i = 1$ for some $i$. It is more hard to deal with all cases analytically, but for $d = 2$ the behavior can be caught numerically. If $L = (1,1)$, then the behavior is the same, but $\text{UP}_{G_G}^d(\tilde{p}_n)$ grows linearly (not exponentially as in item 1)). Two latter cases are plotted in Fig. 1, 1st row.

Now, consider item 2). When $L = (1,1)$, numerically the behavior is the same as in Lemma 1. However, if $L = (1,0)$, then $\text{UP}_{G_G}^d(\tilde{t}_n)$ tends to $1/4$ and $\text{UP}_{G_G}^d(\tilde{p}_n)$ grows linearly (not exponentially as in item 1)). Two latter cases are plotted in Fig. 1, 2nd row.

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REFERENCES