Frame Bounds for Gabor Frames in Finite Dimensions

Palina Salanevich
University of California, LA
Email: psalanevich@math.ucla.edu.

***Abstract*—One of the key advantages of a frame compared to a basis is its redundancy. Provided we have a control on the frame bounds, this redundancy allows, among other things, to achieve robust reconstruction of a signal from its frame coefficients that are corrupted by noise, rounding error, or erasures. In this paper, we discuss frame bounds for Gabor frames (g, Λ) with generic frame set Λ and random window g. We show that, with high probability, such frames have frame bounds similar to the frame bounds of randomly generated frames with independent frame vectors.**

I. Introduction

In the finite dimensional setup, we call a finite set of vectors $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ a *frame* with *frame bounds* $0 < A \leq B$ if, for any $x \in \mathbb{C}^M$,

$$A||x||_2^2 \leq \sum_{j=1}^N |\langle x, \varphi_j \rangle|^2 \leq B||x||_2^2.$$

The values $\langle x, \varphi_j \rangle$, $j \in \{1, \ldots, N\}$, are called the *frame coefficients* of x with respect to the frame Φ.

We note that the above inequality holds for some $0 < A \leq B$ if and only if $\operatorname{span}(\Phi) = \mathbb{C}^M$. That is, the notion of a frame is equivalent to the notion of a spanning set of $\mathbb{C}^M$ in the finite dimensional case.

By a slight abuse of notation, we identify a frame Φ with its *synthesis matrix* Φ, having the frame vectors φ_j as its columns. The adjoint Φ^* of the synthesis matrix is called the *analysis matrix* of the frame Φ. To reconstruct a vector from its frame coefficients, one can use a *dual frame* $\tilde{\Phi} = \{\tilde{\varphi}_j\}_{j=1}^N$, defined so that $x = \sum_{j=1}^N \langle x, \varphi_j \rangle \tilde{\varphi}_j$, for each $x \in \mathbb{Z}_M$. A dual frame is not uniquely defined if $|\Phi| > M$. The *standard dual frame* of Φ is given by the Moore-Penrose pseudoinverse $(\Phi\Phi^*)^{-1}\Phi$ of the synthesis matrix Φ. For a complete background on frames in finite dimensions, we refer the reader to [1].

Frames proved to be a powerful tool in many areas of applied mathematics, computer science, and engineering. The investigation of various properties of frames, such as frame bounds, plays a crucial role in many signal processing problems. Among such problems are communication systems, where the frame coefficients are used to transmit a signal over the communication channel; image processing; and also tomography, speech recognition and brain imaging, where the initial signal is not available, but we have access to its measurements in the form of the frame coefficients instead. The redundancy of the signal representation using frame coefficients allows, among other things, to achieve robust reconstruction of a signal from its frame coefficients that are corrupted by noise, rounding error due to quantization, or erasures.

Indeed, consider a frame $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$. The optimal lower and upper frame bounds of Φ are given by

$$A = \min_{x \in \mathbb{S}^{M-1}} \sum_{j=1}^N |\langle x, \varphi_j \rangle|^2 = \min_{x \in \mathbb{S}^{M-1}} ||\Phi\Phi^* x||_2 = \sigma_{\min}^2(\Phi^*),$$

$$B = \max_{x \in \mathbb{S}^{M-1}} \sum_{j=1}^N |\langle x, \varphi_j \rangle|^2 = \max_{x \in \mathbb{S}^{M-1}} ||\Phi\Phi^* x||_2 = \sigma_{\max}^2(\Phi^*),$$

where $\mathbb{S}^{M-1} = \{x \in \mathbb{C}^M, ||x||_2 = 1\}$ denotes the complex unit sphere, and $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$ denote the smallest and the largest singular values of a matrix A, respectively.

Let $c \in \mathbb{C}^N$ be a vector of noisy frame coefficients of a signal $x \in \mathbb{C}^M$ with respect to the frame Φ. That is,

$$c = \Phi^* x + \delta,$$

where $\delta \in \mathbb{C}^N$ is a noise vector. Then an estimate $\tilde{x}$ of the initial signal x can be obtained from its noisy measurements c using the standard dual frame of Φ as

$$\tilde{x} = (\Phi\Phi^*)^{-1}\Phi c = x + (\Phi\Phi^*)^{-1}\Phi\delta.$$

Thus, for the reconstruction error we have

$$||\tilde{x} - x||_2^2 \leq ||(\Phi\Phi^*)^{-1}\Phi||_2^2 ||\delta||_2^2 = \frac{||\delta||_2^2}{\sigma_{\min}^2(\Phi^*)}.$$

Moreover, if we know a bound on the signal to noise ratio $\mathrm{SNR} = \frac{||\Phi^* x||_2}{||\delta||_2}$ for the channel used, then the norm of the reconstruction error $||(\Phi\Phi^*)^{-1}\Phi\delta||_2$ compares to the norm of the initial signal $||x||_2$ as

$$\frac{||(\Phi\Phi^*)^{-1}\Phi\delta||_2}{||x||_2} \leq \frac{\mathrm{Cond}(\Phi^*)}{\mathrm{SNR}},$$

where, $\mathrm{Cond}(\Phi^*) = \frac{\sigma_{\max}(\Phi^*)}{\sigma_{\min}(\Phi^*)} = \frac{\sqrt{B}}{\sqrt{A}}$ is the condition number of the analysis matrix of the frame Φ.

Thus, frame bounds, or extreme singular values of the frame analysis matrix, indicate the "quality" of a frame in the sense of the robustness of the reconstruction of an initial signal from its noisy frame coefficients. In the case when frame bounds of Φ are sufficiently close to each other, that is, when $\mathrm{Cond}(\Phi^*)$ is not too large, we call the frame Φ *well-conditioned*.

Extreme singular values are sufficiently well-studied for random matrices with independent entries, which can be viewed as analysis matrices of randomly generated frames with independent frame vectors. At the same time, the concrete application for which a signal processing problem is studied usually dictates the structure of the frame used to represent a signal. This motivates the study of properties of structured random matrices corresponding to application relevant frames, such as Gabor frames. (Here and in the sequel, $\mathbb{Z}_M = \{0, 1, \ldots, M-1\}$ denotes the additive group of integers modulo M.)

Definition I.1 (Gabor frames)**.**

1) *Translation* (or *time shift*) by $k \in \mathbb{Z}_M$, is given by
$$T_k x = (x(m-k))_{m \in \mathbb{Z}_M}.$$
2) *Modulation* (or *frequency shift*) by $\ell \in \mathbb{Z}_M$ is given by
$$M_\ell x = \left(e^{2\pi i \ell m/M} x(m)\right)_{m \in \mathbb{Z}_M}.$$
3) The superposition $\pi(k,\ell) = M_\ell T_k$ of translation by k and modulation by ℓ is a *time-frequency shift operator*.
4) For $g \in \mathbb{C}^M \setminus \{0\}$ and $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$, the set of vectors
$$(g, \Lambda) = \{\pi(k,\ell) g\}_{(k,\ell)\in\Lambda}$$
is called the *Gabor system* generated by the *window* g and the set Λ. A Gabor system which spans $\mathbb{C}^M$ is a frame and is referred to as a *Gabor frame*.

Here and in the sequel, we view $x \in \mathbb{C}^M$ as a function $x : \mathbb{Z}_M \to \mathbb{C}$, that is, all the operations on indices are done modulo M. A detailed discussion of Gabor frames in finite dimensions and their properties can be found in [5].

The remaining part of this paper is organizes as follows. In Section II we give a brief overview of the results on extreme singular values for random matrices with independent entries. Section III is dedicated to analysis of frame bounds of Gabor frames with random windows. We start by computing frame bounds in the case when Λ has a particular structure and then switch to the case of generic Λ. Finally Section IV contains numerical results supporting the presented results and discussion of further research directions.

II. Related work

Before we study the case when Φ is a Gabor frame, we include here a short overwiew of the results on the singular values for random frames with independent entries. The largest singular value of the analysis matrix Φ^* of a random frame with independent entries can be estimated using Latala's theorem [3].

Theorem II.1. [3] *For $N > M$, let $\Phi^* \in \mathbb{C}^{N\times M}$ be a random matrix whose entries $\varphi_j(m)$ are i. i. d. centered random variables with $\mathrm{Var}(\varphi_j(m)) = \frac{1}{M}$ and $\mathbb{E}\left(|\varphi_j(m)|^4\right) \le \frac{B}{M^2}$ for some constant $B > 1$. Then there exists a constant $C > 0$ depending only on B, such that*
$$\mathbb{E}\left(\sigma_{\max}(\Phi^*)\right) \le C\sqrt{\frac{N}{M}}.$$

The following optimal estimate of the smallest singular value of the analysis matrix for a random subgaussian frame with independent entries is due to Rudelson and Vershynin [6].

Theorem II.2. [6] *For $N > M$, let $\Phi^* \in \mathbb{C}^{N\times M}$ be a random matrix whose entries $\varphi_j(m)$ are i. i. d. centered L-subgaussian random variables with $\mathrm{Var}(\varphi_j(m)) = \frac{1}{M}$. Then, for any $\varepsilon > 0$, there exist constants $C > 0$, $c \in (0,1)$ depend only on L*
$$\mathbb{P}\left\{\sigma_{\min}(\Phi^*) > \varepsilon\left(\sqrt{\tfrac{N}{M}} - \sqrt{\tfrac{M-1}{M}}\right)\right\} \ge 1 - (C\varepsilon)^{N-M+1} + c^N.$$

To the best of our knowledge, singular values of the analysis matrix Φ_Λ^* of a Gabor frame with random window and general Λ, $|\Lambda| > M$, were not studied before. At the same time, the following bounds on the singular values of the synthesis matrix Φ of a Gabor system (g,Λ) with a Steinhaus window g and $|\Lambda| = O\left(\frac{M}{\log(M)}\right)$ have been established in [4].

Theorem II.3. [4] *Let g be a Steinhaus window, that is, $g(j) = \frac{1}{\sqrt{M}} e^{2\pi i y_j}$, $j \in \mathbb{Z}_M$, with y_j independent uniformly distributed on $[0,1)$. Consider a Gabor system (g,Λ) and let $\varepsilon, \delta \in (0,1)$. Suppose further that*
$$|\Lambda| \le \frac{\delta^2 M}{4e(\log(|\Lambda|/\varepsilon) + c)},$$
where $c = \log(e^2/(4(e-1)))$. Then
$$\mathbb{P}\left(1-\delta \le \sigma_{\min}^2(\Phi_\Lambda) \le \sigma_{\max}^2(\Phi_\Lambda) \le 1+\delta\right) \ge 1-\varepsilon.$$

III. Gabor analysis matrices depending on the structure of Λ

Before we discuss the dependence of the frame bounds on the structure and cardinality of Λ, let us consider a simple case when Λ has a particular structure [7]. Analogous result also holds for continuous Gabor frames [2].

Proposition III.1. *Let (g,Λ) be a Gabor system with $\Lambda = F \times \mathbb{Z}_M$, $F \subset \mathbb{Z}_M$, $F \ne \emptyset$, and $g \in \mathbb{C}^M$. Then (g,Λ) is a frame if and only if $\min_{m\in\mathbb{Z}_M}\{||g_{F_m}||_2\} \ne 0$, where g_{F_m} is the restriction of the vector g to the set of coefficients $F_m = \{m-k\}_{k\in F} \subset \mathbb{Z}_M$.*

Moreover, in this case the optimal lower and upper frame bounds for (g,Λ) are $A = M\min_{m\in\mathbb{Z}_M}\{||g_{F_m}||_2^2\}$ and $B = M\max_{m\in\mathbb{Z}_M}\{||g_{F_m}||_2^2\}$, respectively.

Proof. Consider the matrix $\Phi_\Lambda \Phi_\Lambda^*$, where Φ is a synthesis matrix of the Gabor system (g, Λ). For any $m_1, m_2 \in \mathbb{Z}_M$,

$$\begin{aligned}\Phi_\Lambda \Phi_\Lambda^*(m_1, m_2) &= \sum_{\lambda \in \Lambda} (\pi(\lambda)g)(m_1)\overline{(\pi(\lambda)g)(m_2)} \\ &= \sum_{k \in F} g(m_1 - k)\overline{g(m_2 - k)} \sum_{\ell \in \mathbb{Z}_M} e^{2\pi i \ell (m_1 - m_2)/M}.\end{aligned}$$

Since $\sum_{\ell \in \mathbb{Z}_M} e^{2\pi i \ell (m_1 - m_2)/M} = 0$ for $m_1 \neq m_2$, and $\sum_{\ell \in \mathbb{Z}_M} e^{2\pi i \ell (m_1 - m_2)/M} = M$ for $m_1 = m_2$, we obtain

$$\Phi_\Lambda \Phi_\Lambda^*(m_1, m_2) = \begin{cases} 0, & m_1 \neq m_2 \\ M \sum_{k \in F} |g(m_1 - k)|^2, & m_1 = m_2. \end{cases}$$

That is, $\Phi_\Lambda \Phi_\Lambda^* = \operatorname{diag}\{M \sum_{k \in F} |g(m-k)|^2\}_{m \in \mathbb{Z}_M}$ and, thus, $\{\sigma_m(\Phi_\Lambda^*)\}_{m \in \mathbb{Z}_M} = \{\sqrt{M} ||g_{F_m}||_2\}_{m \in \mathbb{Z}_M}$, where $F_m = \{m - k\}_{k \in F} \subset \mathbb{Z}_M$ and g_S denotes the restriction of the vector g to a set of coefficients $S \subset \mathbb{Z}_M$.

In particular, (g, Λ) is a frame if and only if all the diagonal entries of $\Phi_\Lambda \Phi_\Lambda^*$ are nonzero, that is, if and only if $\min_{m \in \mathbb{Z}_M} \{||g_{F_m}||_2\} \neq 0$. Moreover, for the optimal frame bounds A and B we have

$$A = \sigma^2_{\min}(\Phi_\Lambda^*) = \min_{m \in \mathbb{Z}_M} \sigma_m^2(\Phi_\Lambda^*) = M \min_{m \in \mathbb{Z}_M} \{||g_{F_m}||_2\};$$
$$B = \sigma^2_{\max}(\Phi_\Lambda^*) = \max_{m \in \mathbb{Z}_M} \sigma_m^2(\Phi_\Lambda^*) = M \max_{m \in \mathbb{Z}_M} \{||g_{F_m}||_2\}.$$

□

Remark III.2. We note that an analogous result is true for the the case when $\Lambda = \mathbb{Z}_M \times F$, for some $F \subset \mathbb{Z}_M$. Indeed, let $W_M = \frac{1}{\sqrt{M}}\{e^{-2\pi i k \ell/M}\}_{k, \ell \in \mathbb{Z}_M}$ be the normalized Fourier matrix, and consider the Gabor frame (g, Λ') with a window g and $\Lambda' = (-F) \times \mathbb{Z}_M$. Since $W_M M_\ell T_k g = e^{2\pi i k \ell / M} M_{-k} T_\ell W_M g$, we have

$$W_M \Phi_{(g, \Lambda')} \Phi^*_{(g, \Lambda')} W_M^* = \Phi_{(W_M g, \Lambda)} \Phi^*_{(W_M g, \Lambda)}.$$

Then it follows that $\sigma_{\min}(\Phi^*_{(W_M g, \Lambda)}) = \sigma_{\min}(\Phi^*_{(g, \Lambda')})$ and $\sigma_{\max}(\Phi^*_{(W_M g, \Lambda)}) = \sigma_{\max}(\Phi^*_{(g, \Lambda')})$.

Example III.3. Let us now consider several particular classes of random Gabor windows and estimate the frame bounds for the respective Gabor frames with the frame set $\Lambda = F \times \mathbb{Z}_M$ using Proposition III.1.

(i) *Steinhaus window.* Let the window g be chosen so that for each $m \in \mathbb{Z}_M$, $g(m) = \frac{1}{\sqrt{M}} e^{2\pi i y_m}$ and y_m are independent uniformly distributed on $[0, 1)$. Then, for each $m \in \mathbb{Z}_M$, $M \sum_{k \in F} |g(m - k)|^2 = |F|$, and thus $\Phi_\Lambda \Phi_\Lambda^* = |F| I_M$. That is, (g, Λ) is a tight frame with frame bounds $A = B = |F|$.

(ii) *Gaussian window.* For $g \sim \mathcal{CN}\left(0, \frac{1}{M} I_M\right)$, we have, with high probability,

$$\frac{1}{2}|F| < \sigma^2_{\min}(\Phi_\Lambda^*) \leq \sigma^2_{\max}(\Phi_\Lambda^*) < 5|F|. \quad (1)$$

(iii) *Window, uniformly distributed on* $\mathbb{S}^{M-1}$. In the case when the window g is uniformly distributed on the unit sphere $\mathbb{S}^{M-1}$, with high probability,

$$\frac{1}{8}|F| < \sigma^2_{\min}(\Phi_\Lambda^*) \leq \sigma^2_{\max}(\Phi_\Lambda^*) < 20|F|.$$

The examples above show that, in the case when Λ has a regular structure and window g is random, the Gabor frame (g, Λ) has frame bounds that are quite close to each other, and, thus, is well-conditioned.

Let us now turn into consideration of the case when Λ is a general subset of $\mathbb{Z}_M \times \mathbb{Z}_M$. For any $m \in \mathbb{N}$ and a matrix $H = \Phi_\Lambda \Phi_\Lambda^* - \frac{|\Lambda|}{M} I_M$, we use Markov's inequality, the fact that the Frobenius norm majorizes the operator norm, and that H is self-adjoint, to obtain that

$$\begin{aligned}&\mathbb{P}\left\{\frac{|\Lambda|}{M}(1 - \delta) \leq \sigma^2_{\min}(\Phi_\Lambda^*) \leq \sigma^2_{\max}(\Phi_\Lambda^*) \leq \frac{|\Lambda|}{M}(1 + \delta)\right\} \\ &= \mathbb{P}\left\{||H||_2 \leq \frac{|\Lambda|}{M}\delta\right\} = \mathbb{P}\left\{||H||_2^{2m} > \frac{|\Lambda|^{2m}}{M^{2m}}\delta^{2m}\right\} \\ &\leq \frac{M^{2m}}{|\Lambda|^{2m}}\delta^{-2m}\mathbb{E}(||H^m||_2^2) \leq \frac{M^{2m}}{|\Lambda|^{2m}}\delta^{-2m}\mathbb{E}(\operatorname{Tr} H^{2m}).\end{aligned}$$

One can further show that

$$\mathbb{E}\left(\operatorname{Tr} H^m\right) = \sum_{\substack{j_1, j_2, \ldots, j_m \in \mathbb{Z}_M, \\ j_1 \neq j_2 \neq \cdots \neq j_m \neq j_1 \\ (k_1, \ell_1), \ldots, (k_m, \ell_m) \in \Lambda}} e^{\frac{2\pi i}{M} \sum_{t=1}^m \ell_t (j_t - j_{t+1})} E_{\substack{j_1 \ldots j_m \\ k_1 \ldots k_m}},$$

where $E_{\substack{j_1 \ldots j_m \\ k_1 \ldots k_m}} = \frac{1}{M^m}$, if there exists a permutation $\alpha \in S_m$, such that $j_t - k_t = j_{\alpha(t)} - k_{\alpha(t) - 1}$, for all $t \in \{1, \ldots, m\}$; and $E_{\substack{j_1 \ldots j_m \\ k_1 \ldots k_m}} = 0$, otherwise [7]. This translates the problem of finding a bound on the singular values to a combinatorial problem.

Using this idea with $m = 1$, we obtain the following bound on $\sigma^2_{\max}(\Phi_\Lambda^*)$, which does not depend on the structure of Λ but only on its cardinality [7].

Theorem III.4. *Let g be a Steinhaus window. Consider a Gabor system (g, Λ) with $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$. Then, for each fixed $\varepsilon \in (0, 1)$, with probability at least $1 - \varepsilon$,*

$$\sigma^2_{\max}(\Phi_\Lambda^*) \leq \frac{|\Lambda|}{M} + \sqrt{\frac{|\Lambda|}{\varepsilon}\left(1 - \frac{|\Lambda|}{M^2}\right)}.$$

We note that, the bound obtained in Theorem III.4 is tight for a full Gabor frame, when $\Lambda = \mathbb{Z}_M \times \mathbb{Z}_M$. In the case when $|\Lambda| = \alpha M^2$, for some $\alpha \in (0, 1)$, the proven bound gives $\sigma^2_{\max}(\Phi_\Lambda^*) \leq \left(\alpha + \sqrt{\frac{\alpha(1-\alpha)}{\varepsilon}}\right) M$ with probability at least $1 - \varepsilon$, which is similar to the bound for matrices with independent entries obtained in Theorem II.1.

In the following theorem, we consider the case of a randomly selected set Λ. Roughly speaking, this result shows that, for any $\epsilon \in (0, 1)$, a generic subframe (g, Λ)

of $(g, \mathbb{Z}_M \times \mathbb{Z}_M)$ with $|\Lambda| = O(M^{1+\epsilon} \log M)$ is well-conditioned with high probability [7].

Theorem III.5. *Let g be a Steinhaus window. For any fixed $\varepsilon \in (0, \frac{1}{2}]$, consider a Gabor system (g, Λ) with a random set $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$ constructed so that events $\{(k,\ell) \in \Lambda\}$ are independent for all $(k,\ell) \in \mathbb{Z}_M \times \mathbb{Z}_M$ and have probability $\tau = \frac{C \log M}{M^{1-\varepsilon}}$, where $C > 0$ is a sufficiently large constant depending only on ε. Then, with high probability (depending on ε and δ),*

$$\frac{|\Lambda|}{M}(1-\delta) \leq \sigma^2_{\min}(\Phi^*_\Lambda) \leq \sigma^2_{\max}(\Phi^*_\Lambda) \leq \frac{|\Lambda|}{M}(1+\delta).$$

IV. Numerical results and further discussion

In this section we further investigate frame bounds of Gabor frames using numerical simulations. In particular, we aim to numerically analyze the bounds on the extreme singular values of the analysis matrix Φ^*_Λ of a Gabor frame (g, Λ) with a Steinhaus window g in the case when Λ is a random subset of $\mathbb{Z}_M \times \mathbb{Z}_M$.

In this set of numerical simulations, we investigate the behavior of the singular values of the analysis matrix Φ^*_Λ of a Gabor frame (g, Λ) with a Steinhaus window g and set $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$ selected at random, so that $|\Lambda| = O(M)$ with high probability. The obtained numerical results allow to conjecture that, in the case when random Λ is constructed as described in Theorem III.5 with $\tau = \frac{C}{M}$, there exist constants $0 < k < K$ not depending on the ambient dimension M, such that all the singular values of the analysis matrix Φ^*_Λ are inside the interval $\left[k\frac{|\Lambda|}{M}, K\frac{|\Lambda|}{M}\right]$ with high probability, see Figure 1 (top). The bottom of Figure 1 shows the distribution of the singular values of Φ^*_Λ over this interval for the selected dimensions $M = 100, 150, 200, 250, 300$. We formulate the following conjecture.

Conjecture IV.1. *Let g be a Steinhaus window. Consider a Gabor system (g, Λ) with a random set $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$ constructed so that events $\{(k,\ell) \in \Lambda\}$ are independent for all $(k,\ell) \in \mathbb{Z}_M \times \mathbb{Z}_M$ and have probability $\tau = \frac{C}{M}$, where $C > 0$ is a sufficiently large numerical constant. Then, with high probability (depending on $\delta \in (0,1)$),*

$$\frac{|\Lambda|}{M}(1-\delta) \leq \sigma^2_{\min}(\Phi^*_\Lambda) \leq \sigma^2_{\max}(\Phi^*_\Lambda) \leq \frac{|\Lambda|}{M}(1+\delta).$$

In other words, the additional factor of $M^\varepsilon \log M$ in the cardinality of Λ is a side effect of the method used to prove Theorem III.5. Analogous conjectures can be formulated also for other distributions of the Gabor window.

While the results proposed in this paper discuss the case of a generic, randomly generated, Λ, one of the main directions for the future research is to evaluate frame bounds of Gabor frames for all possible frame sets Λ and to investigate their dependencies on the structure of Λ.

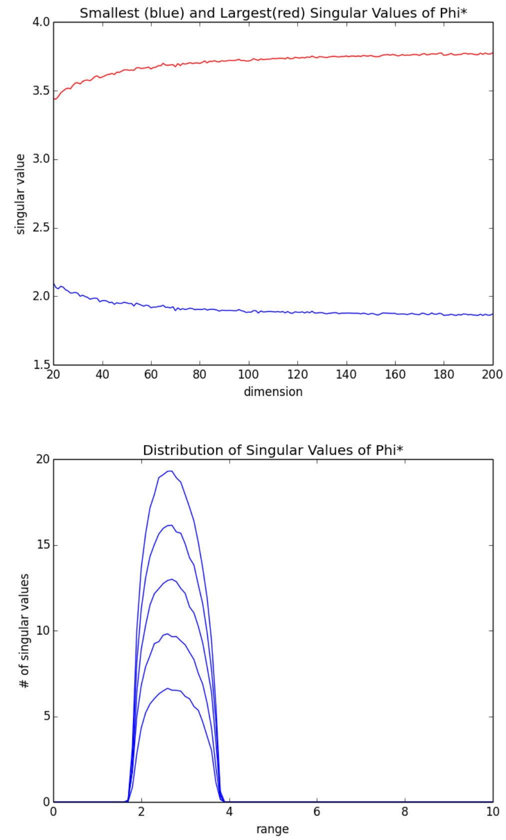

Fig. 1: The top figure shows the dependence of $\sigma^2_{\min}(\Phi^*_\Lambda)$ (blue) and $\sigma^2_{\max}(\Phi^*_\Lambda)$ (red) for a Gabor frame (g, Λ) on the ambient dimension M; and the right hand side of the figure shows the distribution of the singular values of Φ^*_Λ for the dimensions $M = 100, 150, 200, 250, 300$. Here, g is a Steinhaus window and Λ is chosen at random as described in Theorem III.5, with $\tau = \frac{C}{M}$. The number of the numerical experiments is 1000.

V. References

[1] Peter G. Casazza and Gitta Kutyniok. *Finite Frames: Theory and Applications*. Springer, 2013.

[2] Ingrid Daubechies, Alex Grossmann, and Yves Meyer. Painless nonorthogonal expansions. *Journal of Mathematical Physics*, 27(5):1271–1283, 1986.

[3] Rafał Latała. Some estimates of norms of random matrices. *Proceedings of the American Mathematical Society*, 133(5):1273–1282, 2005.

[4] Götz E Pfander and Holger Rauhut. Sparsity in time-frequency representations. *Journal of Fourier Analysis and Applications*, 16(2):233–260, 2010.

[5] Götz E. Pfander. Gabor frames in finite dimensions. In Peter G. Casazza and Gitta Kutyniok, editors, *Finite Frames: Theory and Applications*. Birkhäser Boston, 2013.

[6] Mark Rudelson and Roman Vershynin. Smallest singular value of a random rectangular matrix. *Communications on Pure and Applied Mathematics*, 62(12):1707–1739, 2009.

[7] Palina Salanevich. Extreme singular values of random time-frequency structured matrices. *arXiv preprint arXiv:1902.01062*, 2019.