NP-hardness of ℓ_0 minimization problems: revision and extension to the non-negative setting

Thanh T. NGUYEN[1], Charles SOUSSEN[2], Jérôme IDIER[3], El-Hadi DJERMOUNE[1]
[1]CRAN, Université de Lorraine, CNRS, F-54506 Vandœuvre-lès-Nancy, France
[2]L2S, CentraleSupélec-CNRS-Université Paris-Sud, Université Paris-Saclay, F-91192 Gif-sur-Yvette, France
[3]LS2N, UMR 6004, F-44321 Nantes, France
{thi-thanh.nguyen,el-hadi.djermoune}@univ-lorraine.fr; charles.soussen@centralesupelec.fr; jerome.idier@ls2n.fr

***Abstract*—Sparse approximation arises in many applications and often leads to a constrained or penalized ℓ_0 minimization problem, which was proved to be NP-hard. This paper proposes a revision of existing analyses of NP-hardness of the *penalized* ℓ_0 problem and it introduces a new proof adapted from Natarajan's construction (1995). Moreover, we prove that ℓ_0 minimization problems with non-negativity constraints are also NP-hard.**

I. INTRODUCTION

Sparse approximation appears in a wide range of applications, especially in signal processing, image processing and compressed sensing [1]. Given a signal data $\boldsymbol{y} \in \mathbb{R}^m$ and a dictionary A of size $m \times n$, the aim is to find a signal $\boldsymbol{x} \in \mathbb{R}^n$ that gives the best approximation $\boldsymbol{y} \approx A\boldsymbol{x}$ and has the fewest non-zero coefficients (*i.e.*, sparsest solution). This task leads to solving one of the following constrained or penalized ℓ_0 minimization problems:

$$\min_{\|\boldsymbol{y}-A\boldsymbol{x}\|_2 \le \epsilon} \|\boldsymbol{x}\|_0 \qquad (\ell_0 C)$$

$$\min_{\|\boldsymbol{x}\|_0 \le K} \|\boldsymbol{y}-A\boldsymbol{x}\|_2^2 \qquad (\ell_0 C')$$

$$\min_{\boldsymbol{x}} \|\boldsymbol{y}-A\boldsymbol{x}\|_2^2 + \lambda\|\boldsymbol{x}\|_0 \qquad (\ell_0 P)$$

in which ϵ, K and λ are positive quantities related to the noise standard deviation, the sparsity level and regularization strength, respectively. Letters C and P respectively indicate that the problem is constrained or penalized. Depending on application, the appropriate statement will be addressed. It is noteworthy that n and K often depend on m when one considers the size of problem. ($\ell_0 C$) and ($\ell_0 C'$) are well known to be NP-hard [2, 3]. The NP-hardness of ($\ell_0 P$) was claimed to be a particular case of more general complexity analyses in [4, 5]. However, we point out that these complexity analyses do not rigorously apply to ($\ell_0 P$) as claimed. In this paper, we justify the complexity analyses in [4, 5] do not apply to problem ($\ell_0 P$), and we provide a new proof for the NP-hardness of ($\ell_0 P$) adapted from Natarajan's construction [2].

In several applications such as geoscience and remote sensing [6, 7], audio [8], chemometrics [9] and computed tomography [10], the signal or image of interest is non-negative. In such contexts, one often addresses a minimization problem with both sparsity and non-negativity constraints [10–12]. Adding non-negativity constraints to ℓ_0 minimization problems yields the following problems:

$$\min_{\boldsymbol{x}\ge 0,\, \|\boldsymbol{y}-A\boldsymbol{x}\|_2 \le \epsilon} \|\boldsymbol{x}\|_0 \qquad (\ell_0 C+)$$

$$\min_{\boldsymbol{x}\ge 0,\, \|\boldsymbol{x}\|_0 \le K} \|\boldsymbol{y}-A\boldsymbol{x}\|_2^2 \qquad (\ell_0 C'+)$$

$$\min_{\boldsymbol{x}\ge 0} \|\boldsymbol{y}-A\boldsymbol{x}\|_2^2 + \lambda\|\boldsymbol{x}\|_0 \qquad (\ell_0 P+)$$

Several papers address non-negative ℓ_0 minimization problems in the literature (see, *e.g.*, [13–16]). However, to the best of our knowledge, the complexity of these problems has not been addressed yet, the question of their NP-hardness being still open. Here we show that these problems are NP-hard and the proof can be derived from the NP-hardness of ℓ_0 minimization problems.

The rest of paper is organized as follows. In Section II, we discuss the issues related to NP-hardness of ($\ell_0 P$) in existing analyses and we present our proof. In Section III, we discuss about the NP-hardness of non-negative ℓ_0 minimization problems. We draw some conclusions in Section IV.

II. HARDNESS OF ℓ_0 MINIMIZATION PROBLEMS

A. Background on constrained ℓ_0 minimization problems

Let us recall that an NP-complete problem is a problem in NP to which any other problem in NP can be reduced in polynomial time. Thus NP-complete problems are identified as the hardest problems in NP. An NP-complete problem is strongly NP-complete if it remains NP-complete when all of its numerical parameters are bounded by a polynomial in the length of the input. NP-hard problems are at least as hard as NP-complete problems. However, NP-hard problems do not need to be in NP and do not need to be decision problems. Formally, a problem is NP-hard (respectively, strongly NP-hard) if a NP-complete (respectively, strongly NP-complete) problem can be reduced in polynomial time to it. The reader is referred to [17, 18] for more information on this topic.

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In the literature, problem (ℓ_0C), called SAS in [2], is well known to be NP-hard [2, Theorem 1]. The NP-hardness of (ℓ_0C) is a valuable extension of an earlier result: the problem of minimum weight solution to linear equations (equivalent to (ℓ_0C) with $\epsilon = 0$) is NP-hard [17, p. 246]. Davis *et al.* proved that (ℓ_0C'), called M-optimal approximation in [3], is NP-hard for any $K < m$ [3, Theorem 2.1]. Both analyses of Natarajan and Davis were made by a polynomial time reduction from the "exact cover by 3-sets" problem[1] which is known to be NP-complete [17, p. 221].

B. Existing analyses on penalized ℓ_0 minimization

In [4, 5], the NP-hardness of (ℓ_0P) is deduced as a particular case of more general complexity analyses. However, it turns out that the latter do not apply to (ℓ_0P), as explained hereafter. Chen *et al.* [4] address the unconstrained ℓ_q-ℓ_p minimization problem, defined by:

$$\min_{\boldsymbol{x}} \|\boldsymbol{y} - A\boldsymbol{x}\|_q^q + \lambda\|\boldsymbol{x}\|_p^p \qquad (\ell_q\text{-}\ell_p)$$

where $\lambda > 0$, $q \geq 1$ and $0 \leq p < 1$. The authors showed that problem (ℓ_q-ℓ_p) is NP-hard with any $\lambda > 0$, $q \geq 1$ and $0 \leq p < 1$ [4, Theorem 3]. Obviously, (ℓ_0P) is the case where $q = 2$ and $p = 0$. The proof was done by i) introducing an invertible transformation which scales any instance of problem (ℓ_q-ℓ_p) to the problem (ℓ_q-ℓ_p) with $\lambda = 1/2$, and ii) establishing a polynomial time reduction from the partition problem which is known to be NP-complete [17] to the problem (ℓ_q-ℓ_p) with $\lambda = 1/2$. In other words, they showed that problem (ℓ_q-ℓ_p) with $\lambda = 1/2$ is NP-hard and, because there exists an invertible transformation from any problem (ℓ_q-ℓ_p) to the one with $\lambda = 1/2$, every problem (ℓ_q-ℓ_p) is NP-hard. Similarly, they showed that (ℓ_q-ℓ_p) is strongly NP-hard [4, Theorem 5] by a reduction from the 3-partition problem which is known to be strongly NP-hard [17]. The invertible transform used in [4] is defined by:

$$\tilde{\boldsymbol{x}} = (2\lambda)^{1/p}\boldsymbol{x}, \quad \tilde{A} = (2\lambda)^{-1/p}A. \tag{1}$$

Unfortunately, (1) is not well-defined when $p = 0$. Therefore, [4, Theorems 3 and 5] do not apply to (ℓ_0P) when $\lambda \neq 1/2$.

Using a different approach, Huo and Chen's paper [5] addresses the penalized least-squares problem defined by:

$$\min_{\boldsymbol{x}} \|\boldsymbol{y} - A\boldsymbol{x}\|_2^2 + \lambda\sum_{i=1}^{n} \phi(|x_i|), \qquad \text{(PLS)}$$

where ϕ is a penalty function mapping non-negative values to non-negative values. The authors showed that (PLS) is NP-hard if the penalty function ϕ satisfies the following four conditions [5, Theorem 3.1]:

C1. $\phi(0) = 0$ and $\forall\, 0 \leq \tau_1 < \tau_2$, $\phi(\tau_1) \leq \phi(\tau_2)$.

C2. There exists $\tau_0 > 0$ and a constant $d > 0$ such that

$$\phi(\tau) \geq \phi(\tau_0) - d(\tau_0 - \tau)^2$$

for every $0 \leq \tau < \tau_0$.

C3. For the aforementioned τ_0, if $\tau_1, \tau_2 < \tau_0$ then

$$\phi(\tau_1) + \phi(\tau_2) \geq \phi(\tau_1 + \tau_2).$$

C4. For every $0 \leq \tau < \tau_0$,

$$\phi(\tau) + \phi(\tau_0 - \tau) > \phi(\tau_0). \tag{2}$$

The proof of [5, Theorem 3.1] is by a reduction from the NP-complete problem X3C to the decision version of (PLS); this leads to the NP-completeness of the decision version of (PLS) and so the NP-hardness of (PLS) [5, Appendix 1]. The authors claimed that the ℓ_0 penalty function satisfies conditions C1-C4 for $\tau_0 = d = 1$. Therefore, the (PLS) problem with the ℓ_0 penalty function is NP-hard [5, Corollary 3.2]. Unfortunately, it turns out that the ℓ_0 penalty does not fulfill condition C4 as claimed. Indeed, for $\tau = 0$ the strict inequality (2) becomes $\phi(0) > 0$. Besides, in the proof [5, Appendix 1], the inputs of the decision problem are not guaranteed to have rational values. This might also violate the polynomiality of the reduction. Therefore, [5, Theorem 3.1] does not apply to (ℓ_0P).

In [5], the authors also mention an alternate proof of NP-hardness of (ℓ_0P) from Huo and Ni's earlier paper [19] as a special case of their results. In this proof [19, Appendix A.1], the relation between (ℓ_0P) and (ℓ_0C) is established using the principle of Lagrange multiplier. More precisely, the authors introduce an instance of (ℓ_0C) in which ϵ is defined from the minimizer of (ℓ_0P) and argue that solving (ℓ_0P) is equivalent to solving the mentioned instance of (ℓ_0C), which is known to be NP-hard [2]. There are a number of issues in the NP-hardness proof in [19]. For instance, the proposed transformation between (ℓ_0P) and (ℓ_0C) is not a polynomial time reduction. Besides, it is well known that (ℓ_0P) and (ℓ_0C) are not equivalent [20].

C. New analysis on penalized ℓ_0 minimization problems

To prove that a problem T is NP-hard, one must establish a polynomial time reduction (briefly called reduction hereafter) from some known NP-hard or NP-complete problem to T [18]. Roughly speaking, the reduction from a problem T1 to another problem T2 implies that T1 is not harder than T2. Therefore, if there exists a reduction from T1 to T2 and if T1 is NP-hard, T2 must be NP-hard too. The NP-hardness proofs in [2] and [3] use this principle. As an adaptation of Natarajan's construction, we prove the NP-hardness of (ℓ_0P) using the same principle as follows.

Theorem II.1. *Problem (ℓ_0P) is NP-hard for $0 < \lambda < 3$.*

The proof is by a reduction from the known NP-complete problem X3C to (ℓ_0P). The proof contains three steps: (1) Construct an instance of (ℓ_0P) from a given instance of X3C; (2) Construct a solution of (ℓ_0P) from a solution of X3C; (3) Construct a solution of X3C from a solution of (ℓ_0P).

1) Construction of an instance of (ℓ_0P) from a given instance of X3C: Given an instance of X3C: $S = \{s_1, s_2, ..., s_m\}$ is a set of m elements. C is a collection of n triplets c_j, $1 \leq j \leq n$. Without loss of generality we

[1]The latter problem, denoted by X3C in [2, 17], is stated as follows: Given a set S and a collection C of 3-element subsets of S (called triplets), is there a subcollection of disjoint triplets that exactly covers S?

can assume that m is a multiple of 3 since otherwise there is trivially no exact cover so no solution of X3C.

We now construct an instance of $(\ell_0 P)$. Let $\boldsymbol{y} = [1, 1, ..., 1]^T \in \mathbb{R}^m$. Let $A = (a_{ij})_{1\le i\le m, 1\le j\le n}$ where $a_{ij} = 1$ if $s_i \in c_j$ and $a_{ij} = 0$ otherwise. Let $\lambda \in \mathbb{Q}$, $0 < \lambda < 3$. Let

$$F(\boldsymbol{x}) := \|\boldsymbol{y} - A\boldsymbol{x}\|_2^2 + \lambda\|\boldsymbol{x}\|_0. \tag{3}$$

2) Construction of a solution of $(\ell_0 P)$ from a solution of X3C: Assume that there is a subcollection of disjoint triplets $\hat{C}$ which exactly covers S. Let $\boldsymbol{x}^* = [x_1^*, x_2^*, ..., x_n^*]^T$ where $x_j^* = 1$ if $c_j \in \hat{C}$ and $x_j^* = 0$ otherwise. We will prove that $\boldsymbol{x}^*$ is a solution of $(\ell_0 P)$.

Since $\hat{C}$ exactly covers S, $|\hat{C}| = m/3$ and $\boldsymbol{y} = A\boldsymbol{x}^*$. Thus, $\|\boldsymbol{x}^*\|_0 = m/3$ and

$$F(\boldsymbol{x}^*) = 0 + \lambda\frac{m}{3} = \lambda\frac{m}{3}.$$

Suppose that there exists $\bar{\boldsymbol{x}}$ such that

$$F(\bar{\boldsymbol{x}}) < F(\boldsymbol{x}^*) = \lambda\frac{m}{3}. \tag{4}$$

Let us show that this leads to a contradiction.

Since $F(\bar{\boldsymbol{x}}) \ge \lambda\|\bar{\boldsymbol{x}}\|_0$, from (4) we have $\|\bar{\boldsymbol{x}}\|_0 < m/3$. Therefore, we can rewrite $\|\bar{\boldsymbol{x}}\|_0 = m/3 - q$ for some $q \in \mathbb{N}$, $1 \le q < m/3$. Note that $A\bar{\boldsymbol{x}}$ has m entries. Since the number of non-zero entries of $A\bar{\boldsymbol{x}}$ identifies with the number of elements s_i recovered by the subcollection corresponding to $\bar{\boldsymbol{x}}$, this number cannot exceed $3\|\bar{\boldsymbol{x}}\|_0 = m - 3q$. As a result, the number of zero entries of $A\bar{\boldsymbol{x}}$ must be between $3q$ and m. Since $\boldsymbol{y}$ is the all-one vector, $\boldsymbol{y} - A\bar{\boldsymbol{x}}$ has at least $3q$ entries valued 1, which implies

$$\|\boldsymbol{y} - A\bar{\boldsymbol{x}}\|_2^2 \ge 3q. \tag{5}$$

Hence,

$$F(\bar{\boldsymbol{x}}) \ge 3q + \lambda\left(\frac{m}{3} - q\right) = \lambda\frac{m}{3} + (3-\lambda)q > \lambda\frac{m}{3}, \tag{6}$$

which contradicts (4). Therefore, $\boldsymbol{x}^*$ is a solution of $(\ell_0 P)$.

3) Construction of a solution of X3C from a solution of $(\ell_0 P)$: Assume that $\boldsymbol{x}^*$ is a solution of $(\ell_0 P)$. We will consider four cases as follows.

a) Case $\|\boldsymbol{x}^*\|_0 > m/3$: We deduce that X3C has no solution. Indeed, assume that $\hat{C}$ is an exact cover for S. Define $\boldsymbol{x} = [x_1, x_2, ..., x_n]^T$ where $x_j = 1$ if $c_j \in \hat{C}$ and $x_i = 0$ otherwise. Then we have

$$F(\boldsymbol{x}) = \lambda\frac{m}{3} < \lambda\|\boldsymbol{x}^*\|_0 \le F(\boldsymbol{x}^*)$$

which contradicts the fact that $\boldsymbol{x}^*$ is a solution of $(\ell_0 P)$.

b) Case $\|\boldsymbol{x}^*\|_0 < m/3$: We deduce that X3C has no solution. Indeed, assume that $\hat{C}$ is an exact cover for S. Let $\boldsymbol{x} = [x_1, x_2, ..., x_n]^T$ where $x_j = 1$ if $c_j \in \hat{C}$ and $x_i = 0$ otherwise. Then we have $F(\boldsymbol{x}) = \lambda\frac{m}{3}$. Since $\|\boldsymbol{x}^*\|_0 < m/3$, we can write $\|\boldsymbol{x}^*\|_0 = m/3 - q$ for some $q \in \mathbb{N}$ and $1 \le q < m/3$. Similar to (6), we have $F(\boldsymbol{x}^*) > \lambda\frac{m}{3}$. Since $F(\boldsymbol{x}) = \lambda\frac{m}{3}$, we obtain $F(\boldsymbol{x}^*) > F(\boldsymbol{x})$ which contradicts the fact that $\boldsymbol{x}^*$ is a solution of $(\ell_0 P)$.

c) Case where $\|\boldsymbol{x}^*\|_0 = m/3$ and $\boldsymbol{y} \ne A\boldsymbol{x}^*$: We deduce that X3C has no solution. Indeed, assume that $\hat{C}$ is an exact cover for S. Define $\boldsymbol{x} = [x_1, x_2, ..., x_n]^T$ where $x_j = 1$ if $c_j \in \hat{C}$ and $x_i = 0$ otherwise. Then we have

$$F(\boldsymbol{x}) = \lambda\frac{m}{3} < \|\boldsymbol{y} - A\boldsymbol{x}^*\|_2^2 + \lambda\|\boldsymbol{x}^*\|_0 = F(\boldsymbol{x}^*)$$

which contradicts the fact that $\boldsymbol{x}^*$ is a solution of $(\ell_0 P)$.

d) Case where $\|\boldsymbol{x}^*\|_0 = m/3$ and $\boldsymbol{y} = A\boldsymbol{x}^*$: Let $\hat{C}$ be the collection of triplets c_j such that the j^{th} entry of $\boldsymbol{x}^*$ is non-zero. Obviously, $\hat{C}$ is an exact cover for S so a solution of X3C.

Thus Theorem II.1 is proved.

It is notable that the proof above is also valid when $F(\boldsymbol{x}) := \|\boldsymbol{y} - A\boldsymbol{x}\|_p^p + \lambda\|\boldsymbol{x}\|_0$ for any $p \ge 1$. Indeed, one only needs to check whether (5) still holds when the ℓ_2 norm is replaced by the ℓ_p norm with $p \ge 1$. This is the case since $\boldsymbol{y} - A\bar{\boldsymbol{x}}$ has at least $3q$ entries equal to 1. Therefore, we have the following generalization of Theorem II.1.

Theorem II.2. *Problem $\min_{\boldsymbol{x}} \|\boldsymbol{y} - A\boldsymbol{x}\|_p^p + \lambda\|\boldsymbol{x}\|_0$ is NP-hard for $p \ge 1$ and $0 < \lambda < 3$.*

III. Hardness of non-negative ℓ_0 minimization problems

The NP-hardness of non-negative ℓ_0 minimization problems is a consequence of NP-hard proofs of $(\ell_0 C)$ [2], $(\ell_0 C')$ [3] and $(\ell_0 P)$ (Theorem II.1). Indeed, all these proofs consist in a reduction from X3C and the solution that established equivalence is binary. Therefore, the additional non-negativity constraints do not change the validity of these proofs. In other words, one can repeat the same proofs as in [2, 3] and that of Theorem II.1 for the corresponding non-negative ℓ_0 minimization problems $(\ell_0 C+)$, $(\ell_0 C'+)$ and $(\ell_0 P+)$. Another way to prove the NP-hardness of non-negative ℓ_0 minimization problems is by a reduction from the corresponding ℓ_0 minimization problems which are known to be NP-hard. In this reduction, the instance of non-negative problems is defined by

$$\tilde{\boldsymbol{y}} = \boldsymbol{y}, \quad \tilde{A} = [A, -A], \quad \tilde{\boldsymbol{x}} = \begin{bmatrix}\boldsymbol{x}^+\\ \boldsymbol{x}^-\end{bmatrix}$$

where $\boldsymbol{x}^+ = \max\{\boldsymbol{x}, \boldsymbol{0}\}$, $\boldsymbol{x}^- = \max\{-\boldsymbol{x}, \boldsymbol{0}\}$. Naturally, by this construction, one gets $\tilde{\boldsymbol{x}} \ge \boldsymbol{0}$, $\|\tilde{\boldsymbol{x}}\|_0 = \|\boldsymbol{x}\|_0$ and $\tilde{A}\tilde{\boldsymbol{x}} = A\boldsymbol{x}$. The proofs (skipped for brevity) contain three steps similar to that of Theorem II.1.

Therefore, we can state the following theorem without proof.

Theorem III.1. *$(\ell_0 C+)$, $(\ell_0 C'+)$ are NP-hard. The same for $(\ell_0 P+)$ with $0 < \lambda < 3$.*

In the same spirit and using the same argument as at the end of Section II-C one can directly extend Theorem II.2 to the non-negative setting.

Theorem III.2. *Problem $\min_{\boldsymbol{x}\ge\boldsymbol{0}} \|\boldsymbol{y} - A\boldsymbol{x}\|_p^p + \lambda\|\boldsymbol{x}\|_0$ is NP-hard for $p \ge 1$ and $0 < \lambda < 3$.*

IV. Conclusion

NP-hardness of penalized ℓ_0 minimization problems cannot be deduced from previous complexity analyses, as stated in [4, 5]. Here, we introduced a new proof of NP-hardness of penalized ℓ_0 minimization problems when the regularization parameter λ is smaller than 3, by an adaptation of Natarajan's construction [2], while the case $\lambda \geq 3$ is still open. Besides, we showed that the ℓ_0 minimization problems with non-negative constraints are also NP-hard.

This work can be extended in several directions. For instance, researchers interested in what makes NP-hard problems even harder might be interested in the strong NP-hardness of the aforementioned optimization problems. As it is widely believed that X3C is strongly NP-complete, one might easily deduce the strong NP-hardness of $(\ell_0 C)$, $(\ell_0 C')$ and other problems which are reduced from X3C. However, to the best of our knowledge, X3C is only proved to be NP-complete [17, pp. 53, 221] and the strong NP-completeness has not been rigorously shown yet. Therefore, we believe that the question of strong NP-hardness of (non-negative) ℓ_0 minimization problems is not trivial and needs more work in future.

Besides, as (non-negative) ℓ_0 minimization problems are NP-hard, it would be interesting to know if the associated decision problems are in NP (so being NP-complete). Let us consider the decision problem associated with $(\ell_0 C)$: given $\boldsymbol{y} \in \mathbb{Q}^m$, $A \in \mathbb{Q}^{m\times n}$, a positive rational number ϵ and a positive integer K, does there exist $\boldsymbol{x} \in \mathbb{R}^n$ such that $\|\boldsymbol{y} - A\boldsymbol{x}\|_2 \leq \epsilon$ and $\|\boldsymbol{x}\|_0 \leq K$? This decision problem should be in NP since if one can guess a rational solution $\boldsymbol{x}$, it can be verified in polynomial time if $\|\boldsymbol{y} - A\boldsymbol{x}\|_2 \leq \epsilon$ and $\|\boldsymbol{x}\|_0 \leq K$. Similarly, we conjecture that the decision version of other optimization problems mentioned in the paper are in NP as well.

Another perspective is the approximability of aforementioned NP-hard problems. The hardness of approximating $(\ell_0 C)$ was discussed in [21, 22]. It was shown that approximating $(\ell_0 C)$ to within a factor of $(1-\alpha)\ln(n)$, $0 < \alpha < 1$ is NP-hard [22]. Examining whether similar results can be obtained on other NP-hard problems presented in the paper would require more involved theoretical analysis, which is left for future work.

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