Unitarization and Inversion Formula for the Radon Transform for Hyperbolic Motions

Francesca Bartolucci
Università di Genova, Italy
Email: bartolucci@dima.unige.it

Matteo Monti
Università di Genova, Italy
Email: monti.m@dima.unige.it

***Abstract*—Following the version of Helgason's approach considered in [1] based on intertwining properties for irreducible quasi-regular representations, we construct the Radon transform associated to the group of hyperbolic motions of the plane and we obtain a unitarization result and an inversion formula for this Radon transform.**

I. Introduction

It is shown in [1] how representation theory allows treating in a general and unified way the problem of inverting the Radon transform introduced by Helgason [2] associated to dual pairs $(G/K, G/H)$ of homogeneous spaces of a locally compact group G, where K and H are closed subgroups of G.

Precisely, under some technical assumptions, if the quasi-regular representations of G acting on $L^2(G/K)$ and $L^2(G/H)$ are irreducible, then the Radon transform, up to a composition with a suitable pseudo-differential operator, can be extended to a unitary operator intertwining the two representations, see Theorem 1. Such unitarization problem for the Radon transform was already addressed and solved by Helgason in the context of symmetric spaces [3] which, however, does not recover the framework considered in [1]. If, in addition, the representations are square integrable, an inversion formula for the Radon transform based on the voice transform associated to these representations is given.

The above framework collects various types of Radon transforms related to groups of interest in applications. In [1] are illustrated the examples where G is either the similitude group of the plane or the standard shearlet group to which correspond the classical polar Radon transform [4] and the affine Radon transform [5], respectively. Both these groups appear in the classification of four dimensional reproducing subgroups of $Sp(2,\mathbb{R})$ presented in [6] together with the group of hyperbolic motions of the plane.

In this paper we study the Radon transform associated to this group and we give a unitarization result and an inversion formula for this Radon transform based on intertwining properties for irreducible quasi-regular representations, thereby enlarging the list of examples contained in [1]. Furthermore, this Radon transform is seen to be associated to the classical problem of limited angle tomography, where one wants to reconstruct an unknown signal from its integrals over all lines in a limited range of directions.

The paper is organised as it follows. In Section II we recall the notion of Radon transform associated to dual homogeneous spaces and in Section III we present the main results in [1]. Finally, in Section IV we introduce the group of hyperbolic motions and we give an inversion formula for the related Radon transform.

II. Radon transforms between dual homogeneous spaces: an overview

We briefly introduce the notation. We set $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$. The Euclidean norm of a vector $v \in \mathbb{R}^d$ is denoted by $|v|$ and its scalar product with $w \in \mathbb{R}^d$ by $v \cdot w$. For any $p \in [1, +\infty]$ we denote by $L^p(\mathbb{R}^d)$ the Banach space of functions $f\colon \mathbb{R}^d \to \mathbb{C}$ that are p-integrable with respect to the Lebesgue measure $\mathrm{d}x$ and, if $p = 2$, the corresponding scalar product and norm are $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. The Fourier transform is denoted by $\mathcal{F}$ both on $L^2(\mathbb{R}^d)$ and on $L^1(\mathbb{R}^d)$, where it is defined by

$$\mathcal{F}f(\omega) = \int_{\mathbb{R}^d} f(x) \mathrm{e}^{-2\pi i\, \omega\cdot x} \mathrm{d}x, \qquad f \in L^1(\mathbb{R}^d).$$

Let G be a locally compact group, we denote by $L^2(G)$ the Hilbert space of square-integrable functions with respect to a left Haar measure on G. If X is a transitive G-space with origin x_0 and if $g[x]$ denotes the action of G on X, a Borel measure μ of X is relatively invariant if there exists a positive character α of G such that for any measurable set $E \subset X$ and $g \in G$ it holds $\mu(g[E]) = \alpha(g)\mu(E)$. Furthermore, a Borel section is a measurable map $s\colon X \to G$ satisfying $s(x)[x_0] = x$ and $s(x_0) = e$, with e the neutral element of G; a Borel section always exists since G is second countable [7, Theorem 5.11]. If $A \in M_d(\mathbb{R})$, the vector space of square $d\times d$ matrices with real entries, ${}^tA$ denotes its transpose. Finally, we denote the (real) general linear group of size $d \times d$ by $\mathrm{GL}(d,\mathbb{R})$.

A. Dual homogeneous spaces

We consider two transitive G-spaces X and Ξ, where the actions on $x \in X$ and $\xi \in \Xi$ are

$$(g, x) \mapsto g[x], \quad (g, \xi) \mapsto g.\,\xi.$$

We fix $x_0 \in X$ and $\xi_0 \in \Xi$ and we denote by K and H the corresponding stability subgroups, so that $X \simeq G/K$ and $\Xi \simeq G/H$.

We assume that X and Ξ admit relatively invariant measures $\mathrm{d}x$ and $\mathrm{d}\xi$ with positive characters $\alpha\colon G \to (0, +\infty)$ and $\beta\colon G \to (0, +\infty)$, respectively.

The space X is meant to describe the ambient in which the functions to be analyzed live (e.g. the Euclidean plane), while the space Ξ is meant to parametrize the set of submanifolds of X over which one wants to integrate functions (e.g. lines in the plane).

We define the transitive H-space

$$\hat{\xi}_0 = H[x_0] \subset X$$

and we assume that it carries a relatively H-invariant Radon measure $\mathrm{d}m_0$ with positive character $\gamma\colon H \to (0, +\infty)$.

For technical reasons, we suppose that there exists a Borel section $\sigma\colon \Xi \to G$ such that

$$(g, \xi) \mapsto \gamma\big(\sigma(\xi)^{-1} g \sigma(g^{-1}.\,\xi)\big)$$

extends to a positive character of G independent of ξ and we fix it. Observe that the above assumption is implied by the stronger condition $\gamma(\sigma(\xi)) = 1$ for any $\xi \in \Xi$. Finally, we put

$$\hat{\xi} = \sigma(\xi)[\hat{\xi}_0] \subset X, \tag{1}$$

for any $\xi \in \Xi$, which is a closed subset by [2, Lemma 1.1] and we require that the map $\xi \mapsto \hat{\xi}$ is injective to avoid an overlapping parametrisation of the submanifolds of X over which one wants to integrate functions.

This construction can be read in the context of homogeneous spaces in duality in the sense of Helgason [2, Chapter II]. In Helgason's approach all the spaces X, Ξ and ξ_0 are required to carry invariant measures instead of relatively invariant measures. Our weaker assumptions allow for considering a wider variety of cases, such as the similitude group, the standard shearlet group [1] and the group of hyperbolic motions of the plane presented in Section IV.

B. The representations

The group G acts unitarily on $L^2(X, \mathrm{d}x)$ and $L^2(\Xi, \mathrm{d}\xi)$ via the quasi-regular representations defined by

$$\pi(g)f(x) = \alpha(g)^{-1/2} f(g^{-1}[x])$$

and

$$\hat{\pi}(g)F(\xi) = \beta(g)^{-1/2} F(g^{-1}.\,\xi).$$

We assume that both π and $\hat{\pi}$ are irreducible.

C. Radon transform between dual homogeneous spaces

Mimicking Helgason's approach [2], we define the Radon transform of f associated to the homogeneous spaces X and Ξ as the map $\mathcal{R}f\colon \Xi \to \mathbb{C}$ given by

$$\mathcal{R}f(\xi) = \int_{\hat{\xi}} f(x) \mathrm{d}m_\xi(x), \tag{2}$$

where the measure $\mathrm{d}m_\xi$ on $\hat{\xi}$ is defined as the push-forward of $\mathrm{d}m_0$ using the Borel section σ. Note that this depends intrinsically on the choices of $\mathrm{d}m_0$ and σ, and not only on the subset of integration $\hat{\xi}$.

For our purposes, we assume that there exists a non-trivial π-invariant subspace $\mathcal{A}$ of $L^2(X, \mathrm{d}x)$ such that $\mathcal{R}f$ is well defined for all $f \in \mathcal{A}$ and the Radon transform $\mathcal{R}$ from $\mathcal{A}$ into $L^2(\Xi, \mathrm{d}\xi)$ is closable.

III. General results

In this section we recall the main results presented in [1]. The first subsection is devoted to the unitarization problem and the second one to the generalized Radon inversion problem.

A. Unitarization

We recall the main theorem in [1] which may be stated as follows.

Theorem 1. *The Radon transform $\mathcal{R}\colon \mathcal{A} \to L^2(\Xi, \mathrm{d}\xi)$ is a densely defined operator which intertwines the representations π and $\hat{\pi}$ up to a positive character χ of G, namely*

$$\hat{\pi}(g)\mathcal{R}\pi(g)^{-1} = \chi(g)\mathcal{R}, \tag{3}$$

for all $g \in G$, where

$$\chi(g) = \alpha(g)^{1/2}\beta(g)^{-1/2}\gamma(g\sigma(g^{-1}.\xi_0))^{-1}.$$

Furthermore, there exists a unique densely defined positive selfadjoint operator $\mathcal{I}$ in $L^2(\Xi, \mathrm{d}\xi)$ with the property

$$\hat{\pi}(g)\mathcal{I}\hat{\pi}(g)^{-1} = \chi(g)^{-1}\mathcal{I} \tag{4}$$

such that the composite operator $\mathcal{I}\mathcal{R}\colon \mathcal{A} \to L^2(\Xi, \mathrm{d}\xi)$ extends to a unitary operator $\mathcal{Q}\colon L^2(X, \mathrm{d}x) \to L^2(\Xi, \mathrm{d}\xi)$ intertwining π and $\hat{\pi}$, i.e.

$$\hat{\pi}(g)\mathcal{Q}\pi(g)^{-1} = \mathcal{Q}, \qquad g \in G. \tag{5}$$

We refer to [1] for a more detailed formulation of the above theorem and for its proof. The above result is a generalization of Helgason's theorem on the unitarization of the polar Radon transform [2, Theorem 4.1].

B. Inversion formulae

From now on, we require that π is square-integrable. We recall that, under this assumption, there exists $\psi \in L^2(X, \mathrm{d}x)$ such that the voice transform $\mathcal{V}_\psi$

$$(\mathcal{V}_\psi f)(g) = \langle f, \pi(g)\psi \rangle, \qquad g \in G,$$

is an isometry from $L^2(X, \mathrm{d}x)$ into $L^2(G)$ and we have the weakly-convergent reproducing formula

$$f = \int_G (\mathcal{V}_\psi f)(g)\pi(g)\psi \; \mathrm{d}\mu(g), \tag{6}$$

where μ is a Haar measure of G. Since $\mathcal{Q}$ is unitary and satisfies (5), the voice transform reads

$$(\mathcal{V}_\psi f)(g) = \langle \mathcal{Q}f, \hat{\pi}(g)\mathcal{Q}\psi \rangle, \qquad g \in G. \tag{7}$$

Moreover, if we can choose ψ in such a way that $\mathcal{Q}\psi$ is in the domain of the operator $\mathcal{I}$, exploiting property (4) and the selfadjointness of $\mathcal{I}$, we have

$$(\mathcal{V}_\psi f)(g) = \chi(g)\langle \mathcal{R}f, \hat{\pi}(g)\Psi \rangle, \tag{8}$$

for all $f \in \mathcal{A}$, and the reconstruction formula reads

$$f = \int_G \chi(g)\langle \mathcal{R}f, \hat{\pi}(g)\Psi \rangle \pi(g)\psi \; \mathrm{d}\mu(g), \tag{9}$$

where $\Psi = \mathcal{I}\mathcal{Q}\psi$. Equation (9) gives an inversion formula for the Radon transform based on the voice transform associated to the square-integrable representation π, thereby opening the way to new methods for the generalized Radon inversion problem.

IV. The Radon transform for hyperbolic motions

In this section we show that the general results of Section III may be applied to the Radon transform associated to the group of hyperbolic motions of the plane.

A. The homogeneous spaces

We consider the semidirect product $G = \mathbb{R}^2 \rtimes K$, with $K = \{aA_s\Omega_\epsilon \in \mathrm{GL}(2,\mathbb{R}) : a \in \mathbb{R}^*,\ s \in \mathbb{R},\ \epsilon \in \{-1,1\}\}$ where

$$A_s = \begin{bmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{bmatrix}, \qquad \Omega_{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and Ω_1 is the identity matrix. We denote by C_2 the multiplicative group $\{-1,1\}$. Under the identification $K \simeq \mathbb{R}\times\mathbb{R}^*\times C_2$, we write (b,s,a,ϵ) for the elements in G, so that the group law becomes

$$(b,s,a,\epsilon)(b',s',a',\epsilon') = (b + aA_s\Omega_\epsilon b', s+s', aa', \epsilon\,\epsilon').$$

A left Haar measure of G is $\mathrm{d}\mu(b,s,a,\epsilon) = |a|^{-3}\mathrm{d}b\mathrm{d}s\mathrm{d}a\mathrm{d}\epsilon$, where $\mathrm{d}b$, $\mathrm{d}s$ and $\mathrm{d}a$ are the Lebesgue measures on $\mathbb{R}^2$, $\mathbb{R}$ and $\mathbb{R}^*$, respectively and $\mathrm{d}\epsilon$ is the counting measure on C_2.

The group G acts transitively on $X = \mathbb{R}^2$ by the canonical action

$$(b,s,a,\epsilon)[x] = b + aA_s\Omega_\epsilon x, \quad (b,s,a,\epsilon) \in G,\ x \in X. \quad (10)$$

The isotropy at the origin $x_0 = 0$ is the closed subgroup $\{(0,k) : k \in K\} \simeq K$, so that $X \simeq G/K$ and the Lebesgue measure $\mathrm{d}x$ on X is a relatively G-invariant measure with positive character $\alpha(b,s,a,\epsilon) = |a|^2$. It is possible to parametrize lines in the plane, except those with slope -1 or 1, by the space of parameters $\Xi = C_2\times\mathbb{R}\times\mathbb{R}$ as in Figure 1. The group G is a subgroup of affine transformations of the plane and thus maps lines into lines. Its action on this set of lines is given by the formula

$$(b,s,a,\epsilon)^{-1}.(\eta,u,t) = \left(\epsilon\,\eta, u+s, \frac{t-\Omega_\eta w(u)\cdot b}{a}\right),$$

where $w(u) = {}^t(\cosh u, \sinh u)$, and is easily seen to be transitive. The isotropy at $\xi_0 = (1,0,0)$ is

$$H = \{((0,b_2),0,a,1) : b_2 \in \mathbb{R}, a \in \mathbb{R}^*\}.$$

Thus, $\Xi \simeq G/H$. An immediate calculation gives that the measure $\mathrm{d}\xi = \mathrm{d}\eta\mathrm{d}u\mathrm{d}t$, where $\mathrm{d}u$ and $\mathrm{d}t$ are the Lebesgue measures on $\mathbb{R}$ and $\mathrm{d}\eta$ is the counting measure on C_2, is a G-relatively invariant measure on Ξ with positive character $\beta(b,s,a,\epsilon) = |a|$.

Consider now the section $\sigma\colon \Xi \to G$ defined by

$$\sigma(\eta,u,t) = (t\,\Omega_\eta w(-u), -u, 1, \eta).$$

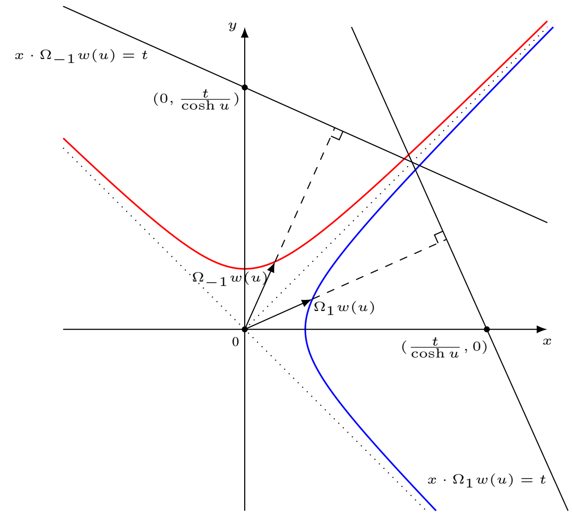

Fig. 1. The lines in $\mathbb{R}^2$ except those with slope 1 or -1 are parametrized by triples $(\eta,u,t) \in \Xi = C_2\times\mathbb{R}\times\mathbb{R}$. The vector u parametrizes the slope. The choice $\eta = 1$ ($\eta = -1$) corresponds to slope > 1 (< 1) and fixes as reference line the $x-$axis (y-axis). Then t parametrizes the intersection of the line with the reference axis.

By direct computation

$$\hat{\xi}_0 = H[x_0] = \{(0,b_2) : b_2 \in \mathbb{R}\} \simeq \mathbb{R}.$$

It is immediate to see that the Lebesgue measure $\mathrm{d}b_2$ on $\hat{\xi}_0$ is a relatively H-invariant measure with character $\gamma((0,b_2),0,a,1) = |a|$ and that $\gamma(\sigma(\eta,u,t)) = 1$ for all $(\eta,u,t) \in \Xi$, so that $(g,\xi) \mapsto \gamma\big(\sigma(\xi)^{-1}g\sigma(g^{-1}.\xi)\big)$ extends to a positive character of G independent of ξ. Further, we have that

$$\widehat{(\eta,u,t)} = \sigma(\eta,u,t)[\hat{\xi}_0] = \{x \in \mathbb{R}^2 : x\cdot\Omega_\eta w(u) = t\},$$

which is the set of all points laying on the line of equation $x\cdot\Omega_\eta w(u) = t$. Therefore, the submanifolds over which we integrate functions are lines in $\mathbb{R}^2$, except to the ones with slope -1 or 1, and are parametrized by Ξ through the injective map $(\eta,u,t) \mapsto \widehat{(\eta,u,t)}$.

B. The representations

The group G acts on $L^2(X)$ by means of the unitary representation π defined by

$$\pi(b,s,a,\epsilon)f(x) = |a|^{-1}f(a^{-1}\Omega_\epsilon^{-1}A_s^{-1}(x-b)).$$

The dual action $\mathbb{R}^2\times K \ni (\eta,k) \mapsto {}^tk\eta$ has a single open orbit $\mathcal{O} = \{(x,y) \in \mathbb{R}^2 : |x| \neq |y|\}$ for ${}^t(1,0) \in \mathbb{R}^2$ of full measure and the stabilizer $K_{(1,0)} = \{(0,1,1)\}$ is compact. Then, by a result due to Führ in [8], the representation π is square-integrable. Furthermore, G acts on $L^2(\Xi,\mathrm{d}\xi)$ by means of the quasi-regular representation $\hat{\pi}$ defined by

$$\hat{\pi}(b,s,a,\epsilon)F(\eta,u,t) = |a|^{-\frac{1}{2}}F\left(\epsilon\,\eta, u+s, \frac{t-\Omega_\eta w(u)\cdot b}{a}\right).$$

By Mackey imprimitivity theorem [9], one can show that also $\hat{\pi}$ is irreducible. The proof, although not trivial, is based on classical arguments and we omit it.

C. The Radon transform

We compute by (2) the Radon transform between the homogeneous spaces X and Ξ and we obtain

$$\mathcal{R}f(\eta,u,t) = \int_{\mathbb{R}} f(\Omega_\eta {A_{-u}}^t(t,y))\mathrm{d}y, \tag{11}$$

which maps any $(\eta,u,t)\in\Xi$ in the integral of f over the line parametrized by (η,u,t) through the map $(\eta,u,t)\mapsto\widehat{(\eta,u,t)}$, i.e. the line of equation $x\cdot\Omega_\eta w(u) = t$. Observe that, by Fubini's theorem, the integral (11) converges for any $f\in L^1(\mathbb{R}^2)$. Then, we define

$$\mathcal{A} = \{f\in L^1\cap L^2(\mathbb{R}^2) : \int_{\mathbb{R}^2}\frac{|\mathcal{F}f(\omega_1,\omega_2)|^2}{\sqrt{|{\omega_1}^2-{\omega_2}^2|}}\mathrm{d}\omega_1\mathrm{d}\omega_2 < +\infty\},$$

which is π-invariant and is such that $\mathcal{R}f\in L^2(\Xi,\mathrm{d}\xi)$ for all $f\in\mathcal{A}$. Furthermore, it is possible to show that $\mathcal{R}$, regarded as operator from $\mathcal{A}$ to $L^2(\Xi,\mathrm{d}\xi)$, is closable. In order to determine the subspace $\mathcal{A}$ and to prove that $\mathcal{R}\colon\mathcal{A}\to L^2(\Xi,\mathrm{d}\xi)$ is closable, we exploit a classical result in Radon theory, the so-called Fourier slice theorem [2, Chapter I], adapted to our context, precisely

$$(I\otimes\mathcal{F})\mathcal{R}f(\eta,u,\tau) = \mathcal{F}f(\tau\,\Omega_\eta w(u)),$$

for every $f\in L^1(\mathbb{R}^2)$ and $(\eta,u,\tau)\in\Xi$, where I is the identity operator on $L^2(C_2\times\mathbb{R},\mathrm{d}\eta\mathrm{d}u)$.

It is worth observing that when we fix $\eta=1$ ($\eta=-1$) in (11) we are restricting the integration of f over all lines with slope >1 (<1). Then, for $\eta=1$ and $\eta=-1$ we have the limited angle horizontal and vertical Radon transforms, respectively. We will see in the next section how these two different contributions enter in the inversion formula when we reconstruct an unknown signal from its Radon transform.

D. Unitarization and Inversion formula

Applying Theorem 1, $\mathcal{R}\colon\mathcal{A}\to L^2(\Xi,\mathrm{d}\xi)$ is a densely defined operator which intertwines the representations π and $\hat{\pi}$ up to the positive character $\chi(b,s,a,\epsilon)=|a|^{-1/2}$, namely

$$\hat{\pi}(b,s,a,\epsilon)\mathcal{R}\pi(b,s,a,\epsilon)^{-1} = |a|^{-1/2}\mathcal{R},$$

for all $(b,s,a,\epsilon)\in G$.

The composition of $\mathcal{R}$ with a positive selfadjoint operator $\mathcal{I}$ satisfying

$$\hat{\pi}(b,s,a,\epsilon)\mathcal{I}\hat{\pi}(b,s,a,\epsilon)^{-1} = |a|^{1/2}\mathcal{I}$$

can be extended to a unitary operator $\mathcal{Q}\colon L^2(\mathbb{R}^2)\to L^2(\Xi,\mathrm{d}\xi)$ intertwining the irreducible representations π and $\hat{\pi}$.

We can provide an explicit formula for $\mathcal{I}$. We consider the subspace $\mathcal{D}$ of $L^2(\Xi,\mathrm{d}\xi)$ of the functions F such that

$$\int_{\mathbb{R}\times\mathbb{R}}|\tau||(I\otimes\mathcal{F})F(\eta,u,\tau)|^2\,\mathrm{d}u\mathrm{d}\tau < +\infty, \qquad \eta=-1,1,$$

and we define the operator $\mathcal{J}$ on $\mathcal{D}$ by

$$(I\otimes\mathcal{F})\mathcal{J}F(\eta,u,\tau) = |\tau|^{\frac{1}{2}}(I\otimes\mathcal{F})f(\eta,u,\tau),$$

a Fourier multiplier with respect to the last variable. A direct calculation shows that $\mathcal{J}$ is a densely defined positive selfadjoint operator with the property

$$\hat{\pi}(b,s,a,\epsilon)\mathcal{J}\hat{\pi}(b,s,a,\epsilon)^{-1} = |a|^{1/2}\mathcal{J}.$$

By [10, Theorem 1], there exists $c>0$ such that $\mathcal{I}=c\mathcal{J}$ and we show that $c=1$.

It is possible to prove that the admissible vectors for π are the functions $\psi\in L^2(\mathbb{R}^2)$ satisfying

$$\int_{\mathbb{R}^2}\frac{|\mathcal{F}\psi(\omega_1,\omega_2)|^2}{|{\omega_1}^2-{\omega_2}^2|}\mathrm{d}\omega_1\mathrm{d}\omega_2 = 1. \tag{12}$$

The voice transform is then $(\mathcal{V}_\psi f)(g)=\langle f,\pi(g)\psi\rangle$, and is a multiple of an isometry from $L^2(\mathbb{R}^2)$ into $L^2(G,\mathrm{d}\mu)$ provided that ψ satisfies the admissible condition (12). If $\mathcal{Q}\psi\in\operatorname{dom}\mathcal{I}$, by equation (8), we have that

$$\begin{aligned}(\mathcal{V}_\psi f)(b,s,a,\epsilon) &= \\ &\int_{\mathbb{R}\times\mathbb{R}}\mathcal{R}f(1,u,t)\overline{\Psi(\epsilon,u+s,\frac{t-w(u)\cdot b}{a})}\frac{\mathrm{d}u\mathrm{d}t}{|a|} \\ &+\int_{\mathbb{R}\times\mathbb{R}}\mathcal{R}f(-1,u,t)\overline{\Psi(-\epsilon,u+s,\frac{t-\Omega_{-1}w(u)\cdot b}{a})}\frac{\mathrm{d}u\mathrm{d}t}{|a|},\end{aligned} \tag{13}$$

for any $f\in\mathcal{A}$, where $\Psi=\mathcal{I}\mathcal{Q}\psi$. Note that the coefficients depend on f only through its Radon transform and do not involve the operator $\mathcal{I}$ as applied to the signal. Hence, equation (6) allows to reconstruct an unknown signal $f\in\mathcal{A}$ from its Radon transform by computing the coefficients $(\mathcal{V}_\psi f)(b,s,a,\epsilon)$ by means of (13). It is worth observing that the different contributions in (13) with $\eta=1$ and $\eta=-1$ reconstruct the frequency projections of f onto the horizontal cone $\{(\omega_1,\omega_2)\in\mathbb{R}^2 : |\omega_2/\omega_1|<1\}$ and onto the vertical cone $\{(\omega_1,\omega_2)\in\mathbb{R}^2 : |\omega_1/\omega_2|<1\}$, respectively. Moreover, if we choose $\Psi(\eta,u,t)=\Psi_2(\eta,u)\Psi_1(t)$, we obtain a formula for the voice transform which involves only integral transforms applied to the Radon transform of the signal, precisely a 1D-wavelet transform, followed by a convolution and it reads

$$\begin{aligned}&(\mathcal{V}_\psi f)(b,s,a,\epsilon) = |a|^{-\frac{1}{2}} \\ &\sum_{\eta=-1,1}(\mathcal{W}_{\Psi_1}(\mathcal{R}f(\eta,\bullet,\cdot))(\Omega_\eta w(\bullet)\cdot b,a) *_u \overline{\Psi_2(\eta\epsilon,\bullet)})(-s),\end{aligned}$$

provided that Ψ_1 is a 1D-wavelet.

Acknowledgment

F. Bartolucci is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

References

[1] G. S. Alberti, F. Bartolucci, F. De Mari, and E. De Vito, "Unitarization and inversion formulae for the Radon transform between dual pairs," *arXiv:1810.12809*, 2018.

[2] S. Helgason, *The Radon transform*, 2nd ed., ser. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1999, vol. 5.

[3] ——, *Geometric analysis on symmetric spaces*, 2nd ed., ser. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2008, vol. 39. [Online]. Available: https://doi.org/10.1090/surv/039

[4] M. Holschneider, "Inverse Radon transforms through inverse wavelet transforms," *Inverse Problems*, vol. 7, no. 6, pp. 853–861, 1991.

[5] F. Bartolucci, F. De Mari, E. De Vito, and F. Odone, "The Radon transform intertwines wavelets and shearlets," *Applied and Computational Harmonic Analysis*, 2018. (available on line https://doi.org/10.1016/j.acha.2017.12.005).

[6] G. S. Alberti, F. De Mari, E. De Vito, and L. Mantovani, "Reproducing subgroups of $Sp(2,\mathbb{R})$. Part II: admissible vectors," *Monatsh. Math.*, vol. 173, no. 3, pp. 261–307, 2014.

[7] V. S. Varadarajan, *Geometry of quantum theory*, 2nd ed. Springer-Verlag, New York, 1985.

[8] H. Führ, "Generalized Calderón conditions and regular orbit spaces," *Colloq. Math.*, vol. 120, no. 1, pp. 103–126, 2010.

[9] G. B. Folland, *A course in abstract harmonic analysis*, 2nd ed., ser. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2016.

[10] M. Duflo and C. C. Moore, "On the regular representation of a nonunimodular locally compact group," *J. Functional Analysis*, vol. 21, no. 2, pp. 209–243, 1976.