On some norms and approximation properties of Kantorovich-type sampling operators

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***Abstract*—The classical sampling theory relies on the exact values of functions taken at some set of points, while in many applications only the local averages in the neighborhood of these points are known. For this reason, we consider the *Kantorovich-type sampling operators* in which the samples are replaced with the average values of a function on a small interval. In this paper, we use the results we have for the classical *generalized sampling operators* to prove the analogous results for the Kantorovich-type sampling operators. In particular, we obtain the exact values of operator norms in L^1 and L^∞, the estimate of operator norm in case of bandlimited kernel as well as the estimate of order of approximation in terms of the modulus of smoothness.**

I. Introduction

For the uniformly continuous and bounded functions $f \in C(\mathbb{R})$, the *generalized sampling operators* are given by ($t \in \mathbb{R}$; $W > 0$)

$$(S_W f)(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) s(Wt-k). \tag{1}$$

The condition for the operator $S_W : C(\mathbb{R}) \to C(\mathbb{R})$ to be well-defined is the kernel $s \in L^1(\mathbb{R})$ satisfying

$$\sum_{k=-\infty}^{\infty} s(u-k) = 1, \qquad \sum_{k=-\infty}^{\infty} |s(u-k)| < \infty \quad (u \in \mathbb{R}), \tag{2}$$

the absolute convergence being uniform on compact intervals of $\mathbb{R}$.

If the kernel function is

$$s(t) = \operatorname{sinc}(t) := \frac{\sin \pi t}{\pi t},$$

we get the classical *Whittaker–Kotelnikov–Shannon operator*

$$(S_W^{\mathrm{sinc}} f)(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) \operatorname{sinc}(Wt-k). \tag{3}$$

A systematic study of sampling operators (1) for arbitrary kernel functions s satisfying (2) was initiated at RWTH Aachen by P. L. Butzer et al. in 1977 (see [1], [2], [3] and references cited there).

The above mentioned sampling operators depend on exact values $f(k/W)$, while in many applications for physical reasons (e.g. the inertia of the measurement apparatus) the results of measurements are some local averages, not the exact point estimates. This situation often occurs in signal processing tasks and is referred to as "time-jitter".

In this case, $f(k/W)$ in (1) can be replaced with an average of f on a small interval around k/W, which gives us the corresponding *Kantorovich-type sampling operators* for locally integrable $f \in C(\mathbb{R})$ and $f \in L^p(\mathbb{R})$, defined by ($t \in \mathbb{R}$; $n \in \mathbb{N}$; $W > 0$)

$$(S_{W,n}^{\mathsf{K}} f)(t) := \sum_{k=-\infty}^{\infty} \left(nW \int_{(2nk-1)/2nW}^{(2nk+1)/2nW} f(u)\, du \right) s(Wt-k). \tag{4}$$

For any $f \in L^p(\mathbb{R})$ ($1 \leqslant p < \infty$) and any locally integrable $f \in C(\mathbb{R})$, we have, respectively, the uniform convergence (see [4])

$$\left\| f - S_W^{\mathsf{K}} f \right\|_p \to 0 \quad \text{and} \quad \left\| f - S_W^{\mathsf{K}} f \right\|_C \to 0 \quad (W \to \infty),$$

$\|\cdot\|_C$ denoting the supremum norm.

Such type of operators was first introduced by L. Kantorovich for Bernstein polynomials in 1930 and later considered in the context of generalized sampling operators in [4] by P. L. Butzer et al. (namely, in the form (4) with $n = 1$), followed by [5], [6], [7], [8], [9], [10], [11], [12] and references cited there. Conceptually similar average sampling in shift-invariant subspaces was considered in e.g. [13], [14], [15]. Quasi-projection operators of similar structure were studied in e.g. [16].

In operators (4), the local averages are calculated via the convolution with the rectangular function. In [17], we introduced the *generalized Kantorovich-type sampling operators*, where the local averages are calculated via the convolution with an arbitrary function $\chi \in L^1(\mathbb{R})$ ($\int_{-\infty}^{\infty} \chi(u)\, du = 1$). However, in this paper we would like to restrict ourselves to the special case of (4) and examine some of its specific properties more closely, using the results from [17].

The rest of the paper is organized as follows. In Section II, we briefly review such notions as Bernstein classes, modulus of smoothness and Jackson-type inequalities. We rely upon these notions when we formulate and prove our main results.

In Section III.A, we give the estimate of operator norm in case if kernel s is bandlimited. We also give the exact values of operator norms in L^1 and L^∞. In Section III.B, we estimate the order of approximation for operators $S^{\mathsf{K}}_{W,n}$ using the existing estimate for corresponding operators S_W. The summary is given in Section IV.

II. Preliminaries

A. Bernstein classes

The Bernstein class B^p_σ is the class of those bounded functions $f \in L^p(\mathbb{R})$ $(1 \leqslant p \leqslant \infty)$ which can be extended to an entire function $f(z)$ $(z \in \mathbb{C})$ of exponential type $\sigma \geqslant 0$ ([2] or [18], 4.3.1), i.e.,

$$|f(z)| \leqslant e^{\sigma|y|}\|f\|_C \quad (z = x + iy \in \mathbb{C}).$$

The class B^p_σ is a Banach space if one takes the norm of $L^p(\mathbb{R})$.

We have $B^1_\sigma \subset B^p_\sigma \subset B^r_\sigma \subset B^\infty_\sigma$, $1 \leqslant p \leqslant r \leqslant \infty$ ([19], L. 6.6; [2], p. 33).

The Bernstein class plays a crucial role in the famous Whittaker–Kotelnikov–Shannon theorem which states ([2], Th. 6.3a): if $f \in B^p_{\pi W}$, $1 \leqslant p < \infty$, or $f \in B^\infty_\sigma$ for some $0 \leqslant \sigma < \pi W$, then

$$(S^{\mathrm{sinc}}_W f)(t) = f(t).$$

B. Modulus of smoothness

The k-th modulus of smoothness ([20], p. 76) of a function $f \in C(\mathbb{R})$ or $f \in L^p(\mathbb{R})$ $(1 \leqslant p < \infty)$ is defined for any $\delta \geqslant 0$ by

$$\omega_k(f,\delta)_C := \sup_{|h|\leqslant\delta} \|\mathring{\Delta}^k_h f(\cdot)\|_C \tag{5}$$

or

$$\omega_k(f,\delta)_p := \sup_{|h|\leqslant\delta} \|\mathring{\Delta}^k_h f(\cdot)\|_p, \tag{6}$$

respectively, where $\mathring{\Delta}^k_h f(\cdot)$ denotes the central difference ([20], p. 197),

$$\mathring{\Delta}^k_h f(x) := \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} f(x + (k/2 - \ell)h). \tag{7}$$

The modulus of smoothness has the following properties ([20], p. 76; [18], 3.3):

$$\begin{array}{rcll} \omega_k(f,\delta) & \leqslant & 2^{k-r}\omega_r(f,\delta) & \text{for any } r \in \mathbb{N},\ r \leqslant k, \\ \omega_k(f,j\delta) & \leqslant & j^k\omega_k(f,\delta) & \text{for any } j \in \mathbb{N}, \\ \omega_k(f,\lambda\delta) & \leqslant & \lfloor 1+\lambda \rfloor^k \omega_k(f,\delta) & \text{for any } \lambda > 0 \end{array} \tag{8}$$

($\lfloor x \rfloor$ is the largest integer less than or equal to $x \in \mathbb{R}$).

C. Jackson-type inequality

Jackson-type inequality, given below, plays an important role in approximation of functions. In particular, it is used to get the estimates of order of approximation for generalized sampling operators S_W (see e.g. [21]). In our work we build upon these results.

Proposition A (cf. [22], Prop. 2)**.** *Given $f \in C(\mathbb{R}) (\equiv L^\infty(\mathbb{R}))$ or $f \in L^p(\mathbb{R})$ $(1 \leqslant p < \infty)$, there exists $g^*_\sigma \in B^p_\sigma$ $(1 \leqslant p \leqslant \infty)$ and a constant $C_k > 0$ (depending only on $k \in \mathbb{N}$) such that*

$$\|f - g^*_\sigma\|_p \leqslant C_k \omega_k(f, 1/\sigma)_p\,.$$

The proof is based on the well-known Jackson-type theorem when approximations are realized by functions from B^p_σ (see, for instance, [18], 2.6.2, 2.6.3, 5.1.3).

III. Main results

A. Operator norms

Recall that the sampling operator $S_W \ : \ C(\mathbb{R}) \to C(\mathbb{R})$ has the norm

$$\|S_W\| := \sup_{u\in\mathbb{R}} \sum_{k=-\infty}^{\infty} |s(u-k)| < \infty. \tag{9}$$

.

In [17], we gave an L^p norm estimate for the generalized Kantorovich-type sampling operators. In case of particular form (4) which we study in this paper, we have the following result.

Proposition 1 (cf [17])**.** *For every $f \in L^p(\mathbb{R})$ $(1 \leqslant p \leqslant \infty)$ there holds* $(n \in \mathbb{N}; \frac{1}{p} + \frac{1}{q} = 1)$

$$\|S^{\mathsf{K}}_{W,n} f\|_p \leqslant n^{1/p}\|S_W\|^{1/q}\|s\|_1^{1/p}\|f\|_p.$$

Being a special case, the proof follows from the result in [17].

Now we get a separate estimate of $\|S^{\mathsf{K}}_{W,n} f\|_p$ for the operators with bandlimited kernels, namely, when $s \in B^1_\pi$.

Corollary 1. *For every $f \in L^p(\mathbb{R})$ $(1 \leqslant p \leqslant \infty)$ and $s \in B^1_\pi$ there holds* $(n \in \mathbb{N})$

$$\|S^{\mathsf{K}}_{W,n} f\|_p \leqslant n^{1/p}\|S_W\|\|f\|_p.$$

Proof. By Nikolskii inequality ([23], p. 124 or [19], Th. 6.8) we have for every $s \in B^p_\sigma$ $(1 \leqslant p \leqslant \infty)$

$$\|s\|_p \leqslant \sup_{u\in\mathbb{R}} \left\{ \sum_{k=-\infty}^{\infty} |s(u-k)|^p \right\}^{1/p} \leqslant (1+\sigma)\|s\|_p. \tag{10}$$

Using (10) and (9) in Proposition 1 completes the proof. □

Now we look at the widely used cases of L^1 and L^∞. In those cases we can compute the exact values of the operator norm.

Theorem 1. *The sampling operator $S^{\mathsf{K}}_{W,n} : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ has the norm*

$$\|S^{\mathsf{K}}_{W,n}\|_{1\to 1} := \sup_{\|f\|_1 \leqslant 1} \|S^{\mathsf{K}}_{W,n} f\|_1 = n\|s\|_1.$$

Proof. By Proposition 1 we have the upper bound

$$\|S^{\mathsf{K}}_{W,n}\|_{1\to 1} \leqslant n\|s\|_1.$$

Now we construct the lower bound. Consider for fixed W and n the function

$$g_{W,n}(t) := \begin{cases} 0, & t < \frac{-1}{2nW}, \\ nW, & \frac{-1}{2nW} \leqslant t \leqslant \frac{1}{2nW}, \\ 0, & t > \frac{1}{2nW}. \end{cases}$$

Obviously, $g_{W,n} \in L^1(\mathbb{R})$ and $\|g_{W,n}\|_1 = 1$. Then we have for $n \geqslant 1$

$$\begin{aligned} &\|S^{\mathsf{K}}_{W,n} g_{W,n}\|_1 \\ &= \int_{\mathbb{R}} \left| \sum_{k=-\infty}^{\infty} nW \int\limits_{(2nk-1)/2nW}^{(2nk+1)/2nW} g_{W,n}(u)\, du\, s(Wt-k) \right| dt \\ &\qquad = \int_{\mathbb{R}} nW\, |s(Wt-k)|\, dt = n\|s\|_1. \end{aligned}$$

Taking into account that

$$\|S^{\mathsf{K}}_{W,n}\|_{1\to 1} := \sup_{\|f\|_1 \leqslant 1} \|S^{\mathsf{K}}_{W,n} f\|_1 \geqslant \|S^{\mathsf{K}}_{W,n} g_{W,n}\|_1 = n\|s\|_1$$

we see that the upper and lower bound coincide which completes the proof. □

The following theorem states that in case of $f \in C(\mathbb{R})$ the operator norms of S_W and the corresponding $S^{\mathsf{K}}_{W,n}$ are the same.

Theorem 2. *If the sampling operator $S_W : C(\mathbb{R}) \to C(\mathbb{R})$ has the finite norm $\|S_W\|$, then the corresponding Kantorovich-type operator $S^{\mathsf{K}}_{W,n} : C(\mathbb{R}) \to C(\mathbb{R})$ has the norm*

$$\|S^{\mathsf{K}}_{W,n}\| := \sup_{\|f\|_\infty \leqslant 1} \|S^{\mathsf{K}}_{W,n} f\|_\infty = \|S_W\|.$$

Proof. By Proposition 1 we have the upper bound

$$\|S^{\mathsf{K}}_{W,n}\| \leqslant \sup_{u\in\mathbb{R}} \sum_{k=-\infty}^{\infty} |s(u-k)|.$$

The series on the right defines a continous function with period one. Therefore,

$$\|S^{\mathsf{K}}_{W,n}\| \leqslant \sup_{-\frac{1}{2}\leqslant u \leqslant \frac{1}{2}} \sum_{k=-\infty}^{\infty} |s(u-k)|. \tag{11}$$

The series on the right is uniformly convergent and for $s \in C(\mathbb{R})$ there exists $u^* \in [-\frac{1}{2}, \frac{1}{2}]$ such that

$$\sup_{-\frac{1}{2}\leqslant u \leqslant \frac{1}{2}} \sum_{k=-\infty}^{\infty} |s(u-k)| = \sum_{k=-\infty}^{\infty} |s(u^*-k)|. \tag{12}$$

Now we construct the lower bound. Consider for fixed W and t the function

$$\begin{aligned} &g_{W,\varepsilon,t}(u) \\ &:= \begin{cases} \frac{2Wu-2k+1}{2\varepsilon} \operatorname{sgn} s(Wt-k), & \frac{2k-1}{2W} \leqslant u < \frac{2k-1+2\varepsilon}{2W}, \\ \operatorname{sgn} s(Wt-k), & \frac{2k-1+2\varepsilon}{2W} \leqslant u \leqslant \frac{2k+1-2\varepsilon}{2W}, \\ \frac{2k+1-2Wu}{2\varepsilon} \operatorname{sgn} s(Wt-k), & \frac{2k+1-2\varepsilon}{2W} < u \leqslant \frac{2k+1}{2W}. \end{cases} \end{aligned}$$

Obviously, $g_{W,\varepsilon,t} \in C(\mathbb{R})$ and $\|g_{W,\varepsilon,t}\|_\infty = 1$. Then we have for $n > 1$ and $\varepsilon < 1/(4nW)$

$$\begin{aligned} &(S^{\mathsf{K}}_{W,n} g_{W,\varepsilon,t})(t) \\ &= \sum_{k=-\infty}^{\infty} \left(nW \int\limits_{(2nk-1)/2nW}^{(2nk+1)/2nW} g_{W,\varepsilon,t}(u)\, du \right) s(Wt-k) \\ &\qquad = \sum_{k=-\infty}^{\infty} |s(Wt-k)| \end{aligned}$$

and for $n = 1$ and $\varepsilon < 1/(4nW)$

$$\begin{aligned} &(S^{\mathsf{K}}_{W,n} g_{W,\varepsilon,t})(t) \\ &= \sum_{k=-\infty}^{\infty} \left(nW \int\limits_{(2nk-1)/2nW}^{(2nk+1)/2nW} g_{W,\varepsilon,t}(u)\, du \right) s(Wt-k) \\ &\qquad = (1-\varepsilon) \sum_{k=-\infty}^{\infty} |s(Wt-k)|. \end{aligned}$$

If we take t^* such that $Wt^* = u^*$ from (12), then for $n > 1$ and $\varepsilon < 1/(4nW)$

$$\|S^{\mathsf{K}}_{W,n}\| \geqslant |(S^{\mathsf{K}}_{W,n} g_{W,\varepsilon,t^*})(t^*)| = \sum_{k=-\infty}^{\infty} |s(u^*-k)| \tag{13}$$

and for $n = 1$ and $0 < \varepsilon < 1/(4nW)$

$$\|S^{\mathsf{K}}_{W,n}\| \geqslant (1-\varepsilon) \sum_{k=-\infty}^{\infty} |s(u^*-k)| \tag{14}$$

which together with (11) and (12) completes the proof. □

B. Order of approximation

In this section, we estimate the order of approximation for the Kantorovich-type sampling operators $S^{\mathsf{K}}_{W,n} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ $(1 \leqslant p \leqslant \infty)$ using the existing estimate for the corresponding operators S_W.

As a first step, we represent the Kantorovich-type sampling operators in terms of Steklov functions (see [18])

$$f_h(t) := h \int\limits_{-1/(2h)}^{1/(2h)} f(t+u)\, du \tag{15}$$

in form

$$(S^{\mathsf{K}}_{W,n} f)(t) = \sum_{k=-\infty}^{\infty} f_{nW}\left(\frac{k}{W}\right) s(Wt-k). \tag{16}$$

Now we are ready to formulate our result.

Theorem 3. *If we can estimate the order of approximation by the operator S_W via the modulus of smoothness of order $r \geqslant 2$, then we have for the corresponding Kantorovich-type operator $S^{\mathsf{K}}_{W,n} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ $(1 \leqslant p \leqslant \infty)$ the estimate*

$$\|f - S^{\mathsf{K}}_{W,n} f\|_p \leqslant M\omega_2(f, 1/W)_p.$$

Proof. We use the representation (16) and have the following estimate

$$\begin{aligned}\|S^{\mathsf{K}}_{W,n} f - f\|_p &= \|S_W f_{nW} - f\|_p \\ &\leqslant \|S_W f_{nW} - f_{nW}\|_p + \|f_{nW} - f\|_p. \quad (17)\end{aligned}$$

We have for $f \in L^p(\mathbb{R})$ also $f_{nW} \in L^p(\mathbb{R})$. If we have the estimate

$$\|S_W f - f\|_p \leqslant M_1 \omega_r \left(f, 1/W\right)_p \quad (r \geqslant 2),$$

then

$$\|S_W f_{nW} - f_{nW}\|_p \leqslant M_1 \omega_r(f_{nW}, 1/W)_p.$$

Using the definition (6) of the modulus of smoothness, we get

$$\begin{aligned}\omega_r(f_h, \delta)_p &= \sup_{|v|\leqslant\delta} \|\mathring{\Delta}^k_v h \int_{-1/(2h)}^{1/(2h)} f(\cdot + u)\, du\|_p \\ &= \sup_{|v|\leqslant\delta} \|h \int_{-1/(2h)}^{1/(2h)} \mathring{\Delta}^k_v f(\cdot + u)\, du\|_p \\ &\leqslant h \int_{-1/(2h)}^{1/(2h)} \sup_{|v|\leqslant\delta} \|\mathring{\Delta}^k_v f(\cdot + u)\|_p\, du \\ &= \sup_{|v|\leqslant\delta} \|\mathring{\Delta}^k_v f(\cdot + u)\|_p = \omega_r(f, \delta)_p.\end{aligned}$$

Then

$$\begin{aligned}\|S_W f_{nW} - f_{nW}\|_p &\leqslant M_1 \omega_r(f_{nW}, 1/W)_p \quad (18)\\ &\leqslant M_1 \omega_r(f, 1/W)_p \leqslant M_1 \omega_2(f, 1/W)_p.\end{aligned}$$

By ([18], 3.12.4) we have, using the properties (8) of the modulus of smoothness, the estimate

$$\|f_{nW} - f\|_p \leqslant \omega_2(f, 1/nW)_p \leqslant \left(1 + \frac{1}{n}\right)^2 \omega_2(f, 1/W)_p. \quad (19)$$

Putting the estimates (18) and (19) into (17) completes the proof. □

IV. Summary

In this paper, we focused on the average sampling by means of Kantorovich-type sampling operators. The main contribution of this paper is the study of operator norms: we gave a norm estimate for operators with bandlimited kernels and exact operator norm values in L^1 and L^∞. We also gave an estimate of order of approximation in terms of the modulus of smoothness.

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