Recovery of a class of Binary Images from Fourier Samples

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Abstract—In this paper, we study the recovery of a certain type of shape images modeled as binary-valued images from few Fourier samples. In particular, the boundary of the shapes is assumed to be the zero level-set of a trigonometric curve. This problem was previously considered in [1], and it was shown that such images can be studied withing the framework of signals with finite rate of innovation. In particular, an annihilation filter is introduced to recover the trigonometric boundary curve. It is proved in [1] that 3Λ Fourier samples are sufficient to exactly recover the edges of a binary shape, where Λ is the bandwith of the trigonometric boundary curve. In this paper, we introduce a class of shapes (that include convex shapes as special cases) for which $|2\Lambda + \mathbb{1}|$ Fourier samples are sufficient for exact recovery using the same annihilation filter. Simulation results support our theoretical results.

I. INTRODUCTION

A natural image is commonly composed of multiple shapes and patterns. For various processing of images, one of the directions is to decompose an image into piece-wise smooth regions [2], [3]. Each region can be thought of as a shape. Detection of such shapes has application in a number of areas such that the detection of ellipses in computer vision [4] and cancer cell detection in biomedical imaging [5], [6]. The challenge of shape recovery is demanding in general, as there is a wide diversity in the geometry of the shapes. Besides, the boundaries of image shapes are oftentimes blurred in digital images. This is due to the fact that shape images are not band-limited in the Fourier domain and the sampling procedure in most acquisition devices includes a blurring kernel (commonly known as the point-spread function).

In this paper, we assume a structure on the boundary of the shapes. In particular, we assume that the boundary can be expressed as a trigonometric curve with finite number of Fourier series coefficients. Therefore, the estimation of the shape can be simplified to extracting the Fourier series coefficients. In other words, the shape image can be expressed via a finite number of parameters, that need to be estimated. This is very similar to the framework of signals with finite rate of innovation (FRI) introduced in [7]. This framework was originally developed for recovering streams of Diracs and piecewise polynomial signals using an annihilation filter method (alternatively known as Prony's method [8]). A similar extension to 2D was proposed in [9], however, this extension was not much inline with 2D images. It was shown in [10] that the FRI framework can be adapted to piecewise smooth images, if each region can be represented as a complex analytic function. In particular, [10] assumes shapes, the boundaries of which are the zero level-set of band-limited periodic complex functions. Similar to the 1D FRI signals, an annihilation filter is designed to extract the underlying band-limited function.

In [1], the boundary of shape images are assumed to be expressed by band-limited trigonometric polynomials. The annihilation equations in this case revolve around the partial derivatives of the image. In fact, the derivatives of a binary shape image (which is piece-wise constant) is sparse. The available information about the shape image is assumed to be low-pass Fourier samples. If the bandwidth of the boundary curve is Λ, it is shown in [1] that at least $(1 + \frac{\sqrt{2}}{2})|\Lambda|$ Fourier samples ($\approx 1.7|\Lambda|$) are required for exact recovery. However, the sufficient number of Fourier samples that is proved in [1] is $3|\Lambda|$. It is noteworthy to mention that the necessary bound ($1.7|\Lambda|$) is observed to work in simulation results. This suggests that the proved sufficient bound might not be tight.

In parallel, the recovery of binary shapes from spatial pixels was studied in [11], [12], again within the framework of FRI signals. In these works, the boundary of shapes are assumed to be algebraic curves. The annihilation filter in [12] works with image moments; i.e., the samples shall be first translated into image moments, before being processed by the annihilation filter that yields the boundary curve. In [13], the required number of image moments for exact recovery is improved compared to [12], given that the shape image is convex.

In this paper, we focus on the recovery of a class of binary images (that represent shapes) from few Fourier samples. The shape boundaries are assumed to be the zero level-sets of band-limited functions with an extra constraint on the shape geometry. In simple (and non-rigorous) words, we call (P, Q) an extreme-pair of points, if P and Q lie on the boundary of the shape and the projection of P and Q onto a line in the plane determine the extreme points of the projection of the whole shape onto this line. Our constraint for the binary shapes is the existence of an extreme-pair (P, Q) that the connecting line PQ does not intersect with the shape boundary other than P and Q. Obviously, this constraint is fulfilled by all convex shapes (the property holds for all extreme-pairs). let $\mathbb{1}$ be set $\{(i, j) : i = \pm 1, 0, \ j = \pm 1, 0\}$ and $\Lambda + \Gamma$ means in the Minkowski sum. Here, we show that for such shapes, $|2\Lambda + \mathbb{1}|$ Fourier samples are sufficient for exact recovery of a shape, where Λ is the bandwidth associated with the boundary curve.
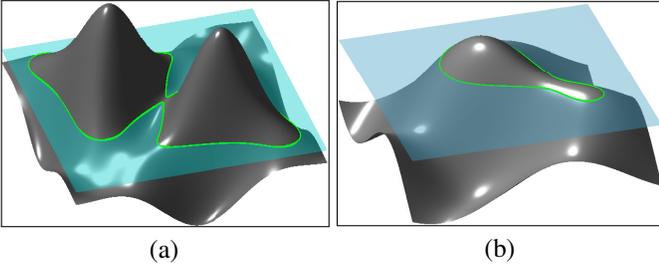
Fig. 1. Example of some trigonometric polynomial and the zero level set. The shapes are the zero level set of a trigonometric curve of 3×3 coefficients.

Fig. 2. FRI recovery framework for piecewise constant image. The partial derivation of image $(\nabla f(r))$ is a stream of Diracs which annihilated by multiplying with the zero level set of a trigonometric polynomial $(\mu(r))$. Approach in Fourier domain.

II. PRELIMINARIES

First, we describe the image model and the annihilation property. Then, we address an existing sampling's theorem.

A. Image Model

Let f be the following 2-D piecewise constant function

$$f(\mathbf{r}) = 1_U(\mathbf{r}), \quad \forall \quad \mathbf{r} = (x, y) \in [0,1]^2, \tag{1}$$

where 1_U is the characteristic function of the region $U \subseteq [0,1]^2$ with piecewise smooth boundary ∂U. In this model, we focus on the Fourier coefficients of the image f which is obtained from the below equation

$$\hat{f}[\mathbf{k}] = \int_{[0,1]^2} f(\mathbf{r})e^{-2j\pi \mathbf{k} \cdot \mathbf{r}}d\mathbf{r}, \quad \mathbf{k} \in \mathbf{\Gamma} \subseteq \mathbb{Z}^2. \tag{2}$$

Where $\mathbf{\Gamma}$ is a finite sampling set. We restrict the edge set to the curves with trigonometric boundaries which can be described as follows:

$$C = \{\mathbf{r} \in [0,1]^2 : \mu(\mathbf{r}) = 0\}, \quad \mu(\mathbf{r}) = \sum_{\mathbf{k} \in \Lambda} c[\mathbf{k}]e^{j2\pi \mathbf{k} \cdot \mathbf{r}} \tag{3}$$

where the coefficients $c[\mathbf{k}] \in \mathbb{C}$, and Λ is any finite subset of $\mathbb{Z}^2$. We call any function μ described by (3) a *trigonometric polynomial*, and its zero level set $C = \{\mu = 0\}$ a *trigonometric curve* which is infinite and has no isolated points. This model is fully compatible with piecewise constant image model and likewise arbitrary image can be estimated by trigonometric curves. Some shapes that have been estimated with low degree trigonometric curves are depicted in Fig.1.

Similar to 1-D finite rate of innovation signals, we have annihilation relation between the derivation of image f and the curve's polynomial. In [1], it was shown that the product of the gradient of f by any smooth periodic function μ vanishing on C is identically zero:

$$\mu \nabla f = 0. \tag{4}$$

Now by taking the Fourier transforms of (4), and by applying the convolution theorem we have

$$\sum_{\mathbf{k} \in \Lambda} c[\mathbf{k}]\widehat{\nabla f}[\ell - k] = 0, \quad \forall \ell \in \mathbb{Z}^2, \tag{5}$$

where $\widehat{\nabla f}[\mathbf{k}] = k(\widehat{\partial_x f}[\mathbf{k}], \widehat{\partial_y f}[\mathbf{k}])$ and also $(c[\mathbf{k}] : \mathbf{k} \in \Lambda)$ are Fourier coefficients of curve μ, called as *annihilating*

polynomial of f. Likewise, $(c[\mathbf{k}] : \mathbf{k} \in \Lambda)$ is called the coefficients of an *annihilating filter*. We write $\Lambda + \Gamma$ in the Minkowski sum $\{k + \ell : \mathbf{k} \in \Gamma, \ell \in \Lambda\}$. Following subsection explains how to estimate the annihilating polynomial using (5) and the edge set from finite low-pass Fourier coefficients of the image.

B. Recovery of edge set from finite Fourier samples

This subsection is dedicated to express the necessary and sufficient condition for recovery of the edge set from Fourier samples of the image using annihilation relation. Likewise, let f be any piecewise constant image. Now, by accessing the Fourier coefficients $\hat{f}[\mathbf{k}]$ for all $\mathbf{k} \in \Gamma$ using (5), we can form the finite linear system of equations as:

$$\sum_{\mathbf{k} \in \Lambda} c[\mathbf{k}]\widehat{\nabla f}[\ell - k] = 0, \quad \forall \ell \in \Gamma : \Lambda, \tag{6}$$

where the index set $\Gamma : \Lambda \subseteq \mathbb{Z}^2$ is the set $\ell \in \mathbb{Z}^2$ such that $\ell - \mathbf{k} \in \Gamma$ for all $\mathbf{k} \in \Lambda$. Now let $\Gamma \subseteq \mathbb{Z}^2$ be an arbitrary sampling set. Also, let the exact Fourier support $\Lambda \subseteq \mathbb{Z}^2$ of the minimal polynomial μ be known. For unique recovery of the Fourier coefficients of μ, $|\Lambda| - 1$ equations are required, as dictated by (6). The two partial derivative gives us $|\Gamma : \Lambda|$ equations and so the necessary condition for the recovery of $c[\mathbf{k}](\mathbf{k} \in \Lambda)$ will be:

$$2|\Gamma : \Lambda| \geq |\Lambda| - 1. \tag{7}$$

For edge set recovery, we rewrite equation (6) in matrix notation as follows:

$$\begin{bmatrix} \mathcal{T}_x(\hat{f}) \\ \mathcal{T}_y(\hat{f}) \end{bmatrix} \mathbf{c} = 0 \tag{8}$$

Where $\mathcal{T}_x(\hat{f}), \mathcal{T}_y(\hat{f}) \in \mathbb{C}^{|\Gamma:\Lambda| \times |\Lambda|}$ are matrices corresponding to the discrete 2-D convolution of the arrays $\widehat{\partial_x f}$ and $\widehat{\partial_y f}$ with annihilation filter $(c[\mathbf{k}] : \mathbf{k} \in \Lambda)$. The annihilating polynomial recovery of image f is equivalent to finding a vector $c[\mathbf{k}]$, which lies in the null space of block Toeplitz matrix $\mathcal{T}(\hat{f})$. In practice, the exact Fourier support is not available. Hence, by focusing on the case of rectangular Fourier supports, following theorem expresses the sufficient condition for the recovery.

Theorem 1 ([1]): Let $f = 1_U$ be the characteristic function of a simply connected region U with boundary $\partial U = \{\mu = 0\}$,
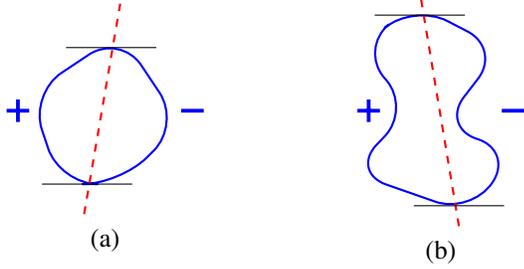
Fig. 3. Shapes derivation with dashed line inside shape that passes through tangent points.
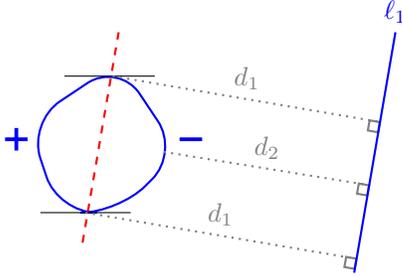
Fig. 4. The derivation of a trigonometric shape image w.r.t. x-axis. Its boundaries separated to positive and negative regions by considering a line inside shape. The top and bottom horizontal line passes through tangent points. ℓ_1 is a line parallel to tangent points and d_1, d_2 are the distances.

where μ is the minimal polynomial. Then, the coefficients $c = (c[\mathbf{k}] : \mathbf{k} \in \Lambda)$ of μ can be uniquely recovered (up to scaling) from samples of $\hat{f}$ in 3Λ, as the only non-trivial solution to the equations

$$\sum_{\mathbf{k} \in \Lambda} c[\mathbf{k}] \widehat{\nabla f}[\boldsymbol{\ell} - k] = 0, \quad \forall \boldsymbol{\ell} \in 2\Lambda, \tag{9}$$

III. MAIN RESULT

According to theorem 1, at least $|3\Lambda|$ Fourier samples are sufficient for recovery of the edge set. However, in [1] experimental simulations show that roughly $1.7|\Lambda|$ samples are sufficient for reconstruction. So, it seems there is a gap between the sufficient and necessary number of samples. In Theorems 2, we decrease the threshold for the required Fourier samples to $|2\Lambda + \mathbb{1}|$ for the class of shapes whose gradient separates to positive and negative regions by a line inside the shape (Fig. 3).

Theorem 2: Let $f = \mathbb{1}_U$ be the characteristic function of a connected region U with boundary $\partial U = \{\mu = 0\}$, where μ is the minimal polynomial. Then, the coefficients $c = (c[\mathbf{k}] : \mathbf{k} \in \Lambda)$ of μ can be uniquely recovered (up to scaling) by $|2\Lambda + \mathbb{1}|$ samples of $\hat{f}$ as the only non-trivial solution to the following equations

$$\sum_{\mathbf{k} \in \Lambda} c[\mathbf{k}] \widehat{\nabla f}[\boldsymbol{\ell} - k] = 0, \quad \forall \boldsymbol{\ell} \in 2\Lambda + \mathbb{1}. \tag{10}$$

Proof 1: We use contradiction. Let $(d[\mathbf{k}])$ be any non-trivial

set of coefficients that can be satisfied in following equation

$$\sum_{\mathbf{k} \in \Lambda} d[\mathbf{k}] \widehat{\nabla f}[\boldsymbol{\ell} - k] = 0, \quad \forall \boldsymbol{\ell} \in 2\Lambda + \mathbb{1}. \tag{11}$$

We construct trigonometric polynomial η with $d[\mathbf{k}] : \mathbf{k} \in \Lambda$ coefficients. Equation (11) represents the Fourier coefficients of $\eta \nabla f$. Due to the restriction of convolution to $2\Lambda + \mathbb{1}$ samples, the spatial domain of (11) ($\eta \nabla f$) is obtained via applying inverse Fourier transform followed by projection onto these set of $2\Lambda + \mathbb{1}$. So, (11) can be written in spatial domain as follows:

$$\mathcal{P}_{2\Lambda}(\eta \nabla f) = 0 \tag{12}$$

Where $\mathcal{P}_{2\Lambda+\mathbb{1}}$ is the Fourier projection onto the index set $2\Lambda + \mathbb{1}$. By applying the inner product ∇f using any test functions of Schwartz class $\mathcal{S}$ (see [14] & [15]) which coincides with all C^∞ smooth complex-valued periodic functions on $[0, 1]^2$, the results will be identically zero. Hence, by considering $\varphi \in \mathcal{S}$ we have:

$$\langle \mathcal{P}_{2\Lambda+\mathbb{1}}(\eta \nabla f), \varphi \rangle = \langle \eta \nabla f, \mathcal{P}_{2\Lambda+\mathbb{1}} \varphi \rangle = 0 \tag{13}$$

or, equivalently,

$$\langle \eta \nabla f, \psi \rangle = 0, \quad \forall \quad \psi \in B^2_{2\Lambda+\mathbb{1}}, \tag{14}$$

where $B^2_{2\Lambda+\mathbb{1}}$ denotes the space of all smooth vector fields $\varphi = (\varphi_1, \varphi_2)$ whose components φ_1, φ_2 are bandlimited to $2\Lambda + \mathbb{1}$. Now, by using Green's theorem for all $\psi \in B^2_{2\Lambda+\mathbb{1}}$, we have:

$$\langle \eta \nabla f, \psi \rangle = \langle \nabla f, \eta \psi \rangle = - \oint_{\partial U} \eta(\psi \cdot \mathbf{n}) ds = 0. \tag{15}$$

where $\mathbf{n}$ is the unit normal vector outward of the curve ∂U. From Prop.10 in [1], we know the set of points where $\nabla \mu$ is identically zero on ∂U is finite. So, the outward normal vector $\mathbf{n}$ can be computed point-wise from $\mathbf{n}(\mathbf{r}) = \pm \nabla \mu(\mathbf{r})/|\nabla \mu(\mathbf{r})|$ for most of the points $\mathbf{r} \in \partial U$, expect the cases where $\nabla \mu(\mathbf{r}) = 0$. Without loss of generality, assume $\mu < 0$ on U. Since μ is a minimal polynomial, by Prop.11 in [1] it must change the sign across any infinite segment of ∂U. So, $\mu > 0$ on U^C. So, $\mathbf{n}(\mathbf{r}) = \nabla \mu(\mathbf{r})/|\nabla \mu(\mathbf{r})|$ for all $\mathbf{r} \in \partial U$. By setting $\psi = \eta^* \zeta \in B^2_{2\Lambda+\mathbb{1}}$ into (15) we have

$$\oint_{\partial U} \frac{|\eta|^2}{|\nabla \mu(s)|} (\zeta(s) \cdot \nabla \mu(s)) ds = 0, \quad \forall \psi \in B^2_{2\Lambda+\mathbb{1}}. \tag{16}$$

The gradient of the shape contains two zero points known as tangent points. Also, based on the gradient direction, the shape is separated into positive and negative continuous regions. Now, we specify the proper choice of $\zeta(s)$. Considering Fig. 4, we set $\zeta(s) = \ell_1$ outside of the shape and parallel to the tangent points. The distance of any point in negative region (d_2) relative to ℓ_1 is shorter than that of the tangent points (d_1). On the other hand, the distance of all the points of the positive region to ℓ_1 is larger than that of the tangent points. Hence, ℓ_1 magnifies the positive region more than the negative one and we can write:

$$\oint_{\partial U} \frac{|\eta|^2}{|\nabla \mu(s)|} (\zeta(s) \cdot \nabla \mu(s)) ds > 0, \quad \text{for all } \psi \in B^2_{2\Lambda+\mathbb{1}}. \tag{17}$$

Equation (17) is not equal to zero unless the zero level-set of μ contains the boundaries of the shape, which is in contradiction with the assumption of finite intersection. Therefore η must be a scalar multiple of μ.

Finally, we will generalize theorem 2 for the case of unknown bandwidth of the edge set. Suppose that the system (15) is built with a larger coefficient support set $\acute{\Lambda}$ than the minimal support Λ. In this case, the solution of (15) is not necessarily unique and it can be shown that every polynomial $\eta = \mu\gamma$ is a solution. Where γ is polynomial with coefficients in $\acute{\Lambda} : \Lambda$. Because $\{\mu = 0\} \subseteq \{\eta = 0\}$, η is also an annihilating polynomial.

Theorem 3: Let f be as in theorem 2. Suppose μ has coefficients in Λ, and let $\acute{\Lambda}$ be another index set with $\Lambda \subseteq \acute{\Lambda}$. If the coefficients $(d[\mathbf{k}] : \mathbf{k} \in \Lambda)$ of a trigonometric polynomial η are a non-trivial solution of the following equation

$$\sum_{\mathbf{k} \in \acute{\Lambda}} c[\mathbf{k}]\widehat{\nabla f}[\ell - k] = 0, \quad \forall \ell \in \Lambda + \acute{\Lambda} + \mathbb{1}, \qquad (18)$$

then μ divides η, that is $\eta = \mu\gamma$ where γ is another trigonometric polynomial with coefficients supported in $\acute{\Lambda} : \Lambda$.

Proof 2: The proof is similar to the proof of theorem 2, with the only difference we replace the index set $\Lambda + \mathbb{1}$ with $\acute{\Lambda} + \mathbb{1}$. This allows us to make the same choice of ψ_j as in (16), which again implies η to vanish on each ∂U_j. Hence, by Prop.2 in [1] we have $\eta = \mu\gamma$, where $\gamma \in B_{\acute{\Lambda}:\Lambda}$.

IV. SIMULATION RESULTS

In this section, a simulation results are presented. Consider a bi-level convex shape with bandwidth $|\Lambda| = 3 \times 3$ which is depicted in Fig. 5(a). Using the Fourier samples and (8), we estimate the trigonometric curve. Fig. 5(b) and 5(c) show the error of the estimated shape, using fewer than and equal to the require samples for recovery, respectively. For the case of adequate bandwidth, the PSNR between the original image and the reconstructed one is 47.20 dB. Also, Fig. 5(d) shows the normalized RMSE of the recovery versus the number of samples used in reconstruction.

V. CONCLUSION

In this paper, we studied a class of shapes whose boundaries are the zero level set of a trigonometric curve. We decreased the number of required Fourier samples for perfect recovery using annihilation equations. We showed that for a trigonometric shape with bandwidth Λ, up to $|2\Lambda + \mathbb{1}|$ number of Fourier samples are sufficient for exact recovery which is an improvement in comparison to that of the $|3\Lambda|$ samples used in [1]. The result is restricted to a class of shapes whose gradients can be separated into two distinct regions by a line inside the shape.

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(a) $\Lambda = 3 \times 3$ (b) $\Lambda = 4 \times 4$ (c) $\Lambda = 5 \times 5$

(d)
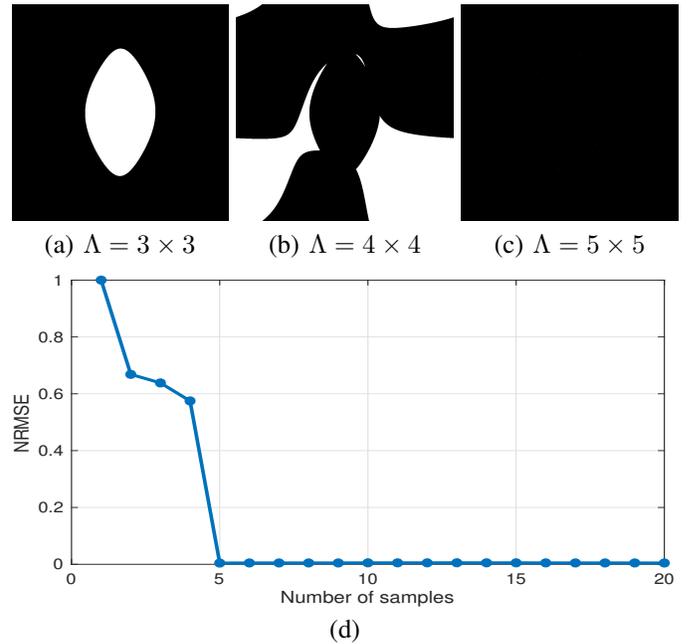
Fig. 5. (a) The continuous image with bandwidth $\Lambda = 3 \times 3$ (b) Error of recovery with filter size $\Lambda = 4 \times 4$ which is below the necessary condition (c) Error of Recovery with filter size $\Lambda = 5 \times 5$ at the necessary condition (PSNR=47.20) (d) draw NRMSE between estimated image and the original image versus number of Fourier samples.

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