Sampling on Hyperbolic Surfaces

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***Abstract*—We discuss harmonic analysis in the setting of hyperbolic space, and then focus on sampling theory on hyperbolic surfaces. We connect sampling theory with the geometry of the signal and its domain. It is relatively easy to demonstrate this connection in Euclidean spaces, but one quickly gets into open problems when the underlying space is not Euclidean. We discuss how to extend this connection to hyperbolic geometry and general surfaces, outlining an *Erlangen*-type program for sampling theory.**

I. INTRODUCTION

This paper discusses harmonic analysis in the setting of hyperbolic space, and then focuses on developing sampling theory on surfaces with an intrinsic hyperbolic geometry.

We start with a brief discussion of sampling in Euclidean space. Section 2 gives an overview of surfaces. This discussion will show the key role of hyperbolic space in general surface theory. It concludes with a discussion of the Uniformization Theorem, which gives that all orientable surfaces inherit their intrinsic geometry from their universal covers. There are only three of these covers – the plane $\mathbb{C}$ (Euclidean geometry), the Riemann sphere $\widetilde{\mathbb{C}}$ (spherical geometry), and the hyperbolic disk $\mathbb{D}$ (hyperbolic geometry). We then develop harmonic analysis in a general setting, looking at the Fourier-Helgason transform and its inversion in the context of Euclidean, spherical, and hyperbolic geometries. Section 3 discusses harmonic Analysis in hyperbolic space, and we conclude with a sampling formula for a hyperbolic surface. This uses the covering theory discussed in Section 2.

There are numerous motivations for extending sampling to non-Euclidean geometries, and in particular, hyperbolic geometry. Irregular sampling of band-limited functions by iteration in hyperbolic space is possible, as shown by Feichtinger and Pesenson and Christensen and Ólafsson ([4], [5]). Hyperbolic space and its importance in Electrical Impedance Tomography (EIT) and Network Tomography has been mentioned in several papers of Berenstein et. al. and some methods developed in papers of Kuchment. More details of these applications may be found in [2], [4]. The paper [2] includes applications to network tomography, with emphasis on the internet.

Classical sampling theory applies to functions that are square integrable and band-limited. A function in $L^2(\mathbb{R})$ whose Fourier transform $\widehat{f}(\omega) = \int_{\mathbb{R}} f(t)e^{-2\pi i t\omega}dt$ is compactly supported and has several smoothness and growth properties given in the Paley-Wiener Theorem (see, e.g., [12], [9]). The choice to have 2π in the exponent simplifies certain expressions, e.g., for $f, g \in L^1 \cap L^2(\mathbb{R})$, $\widehat{f}, \widehat{g} \in L^1 \cap L^2(\widehat{\mathbb{R}})$, we have *Plancherel-Parseval* – $\|f\|_{L^2(\mathbb{R})} = \|\widehat{f}\|_{L^2(\widehat{\mathbb{R}})}$ $\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle$. The Paley-Wiener Space $\mathbb{PW}_\Omega$ is defined as $\mathbb{PW}_\Omega = \{f \text{ continuous} : f, \widehat{f} \in L^2, \operatorname{supp}(\widehat{f}) \subset [-\Omega, \Omega]\}$. The Whittaker-Kotel'nikov-Shannon (W-K-S) Sampling Theorem applies to functions in $\mathbb{PW}_\Omega$.

Theorem I.1 (W-K-S Sampling Theorem)**.** *Let* $f \in \mathbb{PW}_\Omega$, $\operatorname{sinc}_T(t) = \sin(\frac{\pi}{T}t)/\pi t$, *and* $\delta_{nT}(t) = \delta(t - nT)$.

1) If $T \leq 1/2\Omega$, *then for all* $t \in \mathbb{R}$,

$$f(t) = T\left(\left[\sum_{n\in\mathbb{Z}} \delta_{nT}\right] \cdot f\right) * \operatorname{sinc}_T(t)\,. \qquad \text{(I.1)}$$

2) If $T \leq 1/2\Omega$ *and* $f(nT) = 0$ *for all* $n \in \mathbb{Z}$, *then* $f \equiv 0$.

II. GEOMETRY OF SURFACES

A surface is a generalization of the Euclidean plane. This section discusses the geometry of surfaces, and, in particular, three important surfaces – the complex plane $\mathbb{C}$, the Riemann sphere $\widetilde{\mathbb{C}}$, and the unit disc $\mathbb{D}$. Background material for this section can be found in many texts, e.g., [7], [8]. We assume our surfaces are connected and orientable. Therefore, we can choose a coordinate system so that differential forms are positive. We consider *Riemann surfaces*, but our discussion carries through to connected and orientable Riemannian manifolds of dimension two. Riemann surfaces allow us to discuss the *Uniformization Theorem*, which gives that all orientable surfaces inherit their intrinsic geometry from their *universal coverings*.

Klein's *Erlangen program* sought to characterize and classify the different geometries on the basis of projective geometry and group theory. Since there is a lot of freedom in projective geometry, due to the fact that its properties do not depend on a metric, projective geometry became the unifying frame of all other geometries. Also, group theory provided a useful way to organize and abstract the ideas of symmetry for each geometry. The different geometries need their own appropriate languages for their underlying concepts, since objects like circles and angles are not preserved under projective transformations. Instead, one could talk about the subgroups and normal subgroups created by the different concepts of each geometry, and use this to create relations between other geometries. The underlying group structure is the group of isometries under which the geometry is invariant. Isometries are functions that preserve distances and angles of all points in the set. A property of surfaces in $\mathbb{R}^3$ is said to be *intrinsic* if it is preserved by isometry, i.e., if it can be determined from any point on the surface. Isometries can be modeled as the groups of symmetries of the geometry. Thus, the hierarchies of the symmetry groups give a way for us to define the hierarchies of the geometries. We explore the groups of isometries for three geometries – Euclidean, spherical, and hyperbolic. We present the Uniformization Theorem, which shows that for connected and orientable surfaces, these are the only intrinsic geometries.

We say that $\widetilde{\mathcal{S}}$ is an *unlimited* covering of $\mathcal{S}$ provided that for every curve γ on $\mathcal{S}$ and every $\widetilde{\zeta} \in \widetilde{\mathcal{S}}$ with $f(\widetilde{\zeta}) = \gamma(0)$, there exists a curve $\widetilde{\gamma}$ on $\widetilde{\mathcal{S}}$ with initial point $\widetilde{\zeta}$ and $f(\widetilde{\gamma}) = \gamma$. The curve $\widetilde{\gamma}$ is called a *lift* of γ. This is generally referred to as the *curve lifting property*, and it follows directly for an unlimited, unramified covering.

Given a point z_0 on a Riemann surface $\mathcal{S}$, we consider all closed curves on $\mathcal{S}$ passing through z_0. We say that any two of these paths are equivalent whenever they are homotopic. The set of these equivalence classes forms a group with the operation of multiplication of equivalence classes of paths. This group is called the *fundamental group of $\mathcal{S}$ based at z_0* and denoted as $\pi_1(\mathcal{S}, z_0)$. Since all Riemann surfaces are path connected, given any two points z_0, z_1 on $\mathcal{S}$, the groups $\pi_1(\mathcal{S}, z_0)$ and $\pi_1(\mathcal{S}, z_1)$ are isomorphic. This allows us to refer to the *fundamental group of $\mathcal{S}$* ($\pi_1(\mathcal{S})$) by picking any base point on $\mathcal{S}$. Note, if $\mathcal{S}$ is simply connected, $\pi_1(\mathcal{S})$ is trivial.

There is an important connection between $\pi_1(\mathcal{S})$ and the smooth unlimited covering spaces $\widetilde{\mathcal{S}}$ of $\mathcal{S}$. If $\widetilde{\mathcal{S}}$ is a smooth unlimited covering space of $\mathcal{S}$, then $\pi_1(\widetilde{\mathcal{S}})$ is isomorphic to a subgroup of $\pi_1(\mathcal{S})$. Conversely, every subgroup of $\pi_1(\mathcal{S})$ determines a smooth unlimited covering corresponding to the space $\widetilde{\mathcal{S}}$. Given that the trivial group is a subgroup of every group, the group of $\pi_1(\mathcal{S})$ determines a simply connected smooth unlimited covering space $\widetilde{\mathcal{S}}$, which is called the *universal cover*, i.e., the universal covering space is the covering space corresponding to the trivial subgroup of $\pi_1(\mathcal{S})$.

Given a connected Riemann surface $\mathcal{S}$ and its universal covering space $\widetilde{\mathcal{S}}$, $\mathcal{S}$ is isomorphic to $\widetilde{\mathcal{S}}/\Gamma$, where the group Γ is isomorphic to the fundamental group of $\mathcal{S}$, $\pi_1(\mathcal{S})$ (see [7], [8]). The corresponding universal covering is simply the quotient map which sends every point of $\widetilde{\mathcal{S}}$ to its orbit under Γ. Thus, the fundamental group of $\mathcal{S}$ determines its universal cover. Moreover, the universal covering is indeed the "biggest" smooth unlimited covering of a connected Riemann surface, in the sense that all other unramified unlimited covering spaces of a Riemann surface can be covered unlimitedly and without ramification by the universal covering of this surface.

The Uniformization Theorem allows us to classify all universal covers of all Riemann surfaces. This in turn allows us to understand the geometry of every Riemann surface. An open Riemann surface is called *hyperbolic* if the maximum principle is not valid. This is equivalent to the existence of a Green's function and a harmonic measure. An open Riemann surface is called *parabolic* if it does not have these properties. Closed Riemann surfaces are *elliptic*.

Theorem II.1 (The Uniformization Theorem)**.** *Let $\mathcal{S}$ be a Riemann surface.*

1.) *Every surface admits a Riemannian metric of constant Gaussian curvature κ.*

2.) *Every simply connected Riemann surface is conformally equivalent to one of the following:*

- **a.)** $\mathbb{C}$ *with Euclidean Geometry (parabolic) – $\kappa = 0$ – with isometries*
$$\left\langle \left\{ e^{i\theta} z + \alpha \right\}, \circ \right\rangle, \text{ where } \alpha \in \mathbb{C} \text{ and } \theta \in [0, 2\pi),$$
- **b.)** $\widetilde{\mathbb{C}}$ *with Spherical Geometry (elliptic) – $\kappa = 1$ – with isometries*
$$\left\langle \left\{ \frac{\alpha z + \beta}{-\overline{\beta} z + \overline{\alpha}} \right\}, \circ \right\rangle, \text{ where } \alpha, \beta \in \mathbb{C} \text{ and } |\alpha|^2 + |\beta|^2 = 1,$$
- **c.)** $\mathbb{D}$ *with Hyperbolic Geometry (hyperbolic) – $\kappa = -1$ – with isometries*
$$\left\langle \left\{ e^{i\theta} \frac{z - \alpha}{1 - \overline{\alpha} z} \right\}, \circ \right\rangle, \text{ where } |\alpha| < 1 \text{ and } \theta \in [0, 2\pi).$$

Proof is given in Farkas and Kra [7], Section IV.6. We may extend Uniformization to orientable Riemannian manifolds of dimension two. In fact, every orientable topological two-real-dimensional manifold with a countable basis for its topology admits a Riemann surface structure [7]. The consequences of the Uniformization Theorem can be stared very succinctly. The only covering surface of Riemann sphere $\widetilde{\mathbb{C}}$ is itself, with the covering map being the identity. The plane $\mathbb{C}$ is the universal covering space of itself, the once punctured plane $\mathbb{C} \setminus \{z_0\}$ (with covering map $\exp(z - z_0)$), and all tori $\mathbb{C}/\Gamma$, where Γ is a parallelogram generated by $z \longmapsto z + n\gamma_1 + m\gamma_2$, $n, m \in \mathbb{Z}$ and γ_1, γ_2 are two fixed complex numbers linearly independent over $\mathbb{R}$. *The universal covering space of every other Riemann surface is the hyperbolic disk* $\mathbb{D}$. This last result demonstrates the importance of hyperbolic space.

III. Harmonic Analysis in Non-Euclidean Domains

In a very general setting, we can discuss a harmonic analysis of a locally compact Hausdorff space X which is acted upon transitively by a locally compact topological group G [11]. (Recall that a *topological group* G is a group equipped with a topology such that multiplication and inversion are continuous maps.) We will assume that X has a positive measure μ, and that G leaves this measure invariant, e.g., μ is Lebesgue measure on $\mathbb{R}$, and G is the set of normalized linear translations. If G is also abelian, we define a *character* as a continuous group homomorphism $\varphi : G \to \mathbb{T}$. The set of characters form a group $\widehat{G}$ under pointwise multiplication, the dual group. Let $f \in L^1(G)$. The *Fourier transform* is the mapping $\widehat{f} : \widehat{G} \to \mathbb{C}$ defined by $\widehat{f}(\varphi) = \int_G f(x)\overline{\varphi(x)}d\mu(x)$. (In $\mathbb{R}$, this gives $\widehat{f}(\omega) = \int_{\mathbb{R}} f(x)e^{-ix\omega}dx$.)

Let $f \in L^2(X)$ and $g \in G$, and define $T_X(g)f(x)$ as $T_X(g)f(x) = f(g^{-1} \cdot x)$. Then T_X is the unitary representation of G acting on $L^2(X)$. (Recall that a *representation* of G is a pair (T, H), where H is a separable Hilbert space and

$T : G \to \mathbf{GL}(H)$, where $\mathbf{GL}(H)$ is the group of invertible linear maps on H.) The representation is *unitary* if it preserves the inner product. It is called *irreducible* if there is no closed proper subspace W of H such that $T(g)W \subset W$ for all $g \in G$.) A *harmonic analysis* of X is the decomposition of T_X into irreducible elements [10], [11].

Recall that a *Lie group* is a locally Euclidean topological group whose group operations are C^∞ maps. Let us add more structure to X by letting $X = G/K$ be a symmetric space, where the group of symmetries of X contains an inversion symmetry about every point. Symmetric spaces are Riemannian manifolds whose curvature tensor is invariant under all parallel transports. This holds if and only if each geodesic symmetry $\gamma(s) \to \gamma(-s)$ at a point x is a local isometry. For this to hold globally, the space must possess a transitive group K of isometries. Then, $X = G/K$ is a homogeneous space of a Lie group G, where K is a compact subgroup of G, and where the Lie algebra of K is an involution of the Lie algebra of G. We can now use the machinery of the Lie theory. Let $\mathcal{E}$ denote the set of C^∞ functions on X, and $\mathcal{D}$ denote the set of C^∞ functions on X which have compact support. We can consider the algebra $\mathbf{D}(G/K)$ of all differential operators on X which are invariant under all translations of cosets xK by $g \in G$, i.e., $\tau(g) : xK \longrightarrow gxK$. A function on X which is an eigenfunction (actually, eigendistribution) of each $D \in \mathbf{D}(G/K)$ is a *joint eigenfunction* of $\mathbf{D}(G/K)$. Let $\varphi : \mathbf{D}(G/K) \longrightarrow \mathbb{C}$ and

$$E_\varphi(X) = \{f \in \mathcal{E}(X) : Df = \varphi(D)f \text{ for all } D \in \mathbf{D}(G/K)\}\,.$$

$E_\varphi(X)$ is called a *joint eigenspace*. Let T_φ be the representation $(T_\varphi(g)f)(x) = f(g^{-1}x)$. T_φ is called an *eigenspace representation*.

A *harmonic analysis* in this setting is the study of the following. First, the decomposition of arbitrary functions on G/K into joint eigenfunctions of $\mathbf{D}(G/K)$. Second, the cataloging of $E_\varphi(X)$, joint eigenspaces of $\mathbf{D}(G/K)$. Third, determining the set of maps φ for which T_φ is irreducible [10], [11]. Given that K is compact, then $X = G/K$ is Riemannian and hence the algebra $\mathbf{D}(G/K)$ contains an elliptic operator, the Laplacian. Therefore, every eigenfunction is an analytic function.

IV. Harmonic Analysis in Hyperbolic Space

Let $z = x + iy \in \mathbb{C}$, with $x = \Re z$, $y = \operatorname{Im} z \in \mathbb{R}$, and let $\mathbb{U} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. $\mathbf{SL}(2,\mathbb{R})$ is the set of 2×2 matrices with determinant one, and $\mathbf{SO}(2) = \{A \in \mathbf{O}(2) : \det A = 1\}$. The action of $\mathbf{SL}(2,\mathbb{R})$ on $\mathbb{U}$ is given by

$$\left\{g(z) = \frac{az+b}{cz+d} : ad-bc = 1\right\}.$$

$\mathbf{SL}(2,\mathbb{R})$ acts transitively on $\mathbb{U}$ with $\mathbf{SO}(2)$ being the stabilizer of i, giving the identification

$$\mathbb{U} = \mathbf{SL}(2,\mathbb{R})/\mathbf{SO}(2)\,.$$

We have that $\mathbb{U}$ has Riemannian metric $ds^2 = y^{-2}(dx^2+dy^2)$ and corresponding Riemannian measure $d\mu = y^{-2}(dx\,dy)$. We normalize the Haar measure dg on $\mathbf{SL}(2,\mathbb{R})$ so that

$$\int_{\mathbb{U}} f(z)\frac{dxdy}{y^2} = \int_{\mathbf{SL}(2,\mathbb{R})} f(g(i))dg\,.$$

The Hilbert space $L^2(\mathbf{SL}(2,\mathbb{R})$ is the space of square integrable functions with the inner product

$$\langle f, g\rangle = \int_{\mathbb{U}} f(z)\overline{g(z)}\frac{dx\,dy}{y^2}\,.$$

The Laplacian Δ on $\mathbb{U}$ is symmetric and given by

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

If $f, \Delta f \in L^2(\mathbb{U})$, then $\langle f, \Delta f\rangle \le \frac{-1}{4}\|f\|^2$. The *Fourier-Helgason* transform is defined by

$$\widehat{f}(s,\rho) = \int_{\mathbb{U}} f(z)\overline{(\operatorname{Im}(k_\rho(z))^s}\frac{dx\,dy}{y^2}\,,$$

for $s \in \mathbb{C}$, $\rho \in \mathbb{T}$, and $k_\rho \in \mathbf{SO}(2)$ being a rotation by angle ρ. For $f \in \mathcal{D}(\mathbb{U})$, this has the inversion, for $t \in \mathbb{R}$,

$$\frac{1}{8\pi^2}\int_{\mathbb{R}}\int_{\mathbb{T}} \widehat{f}((it{+}1/2),\rho)(\operatorname{Im}(k_\rho(z))^{(it+1/2)} t\tanh(\pi t)d\rho\,dt\,.$$

The map $f \longrightarrow \widehat{f}$ extends to an isometry from $(L^2(\mathbb{U}), d\mu)$ onto $(L^2(\mathbb{R}\times\mathbb{T}, (\frac{1}{8\pi^2}t\tanh(\pi t)d\rho\,dt))$.

These provide us the tools for harmonic analysis on $\mathbb{U}$. Analogously, we can develop hyperbolic geometry in the unit disk $\mathbb{D} = \{z \in C : |z| < 1\}$. The mapping $w = T(z) = \frac{z-i}{z+i}$ conformally maps $\mathbb{U}$ to $\mathbb{D}$ and is an isometry from $(\mathbb{U}, ds_{\mathbb{U}})$ to $(\mathbb{D}, ds_{\mathbb{D}})$. The inverse mapping is $z = T^{-1}(w) = -i\frac{z+1}{z-1}$. From the group theoretic viewpoint, $\mathbf{SL}(2,\mathbb{R}) \sim \mathbf{SU}(1,1)$. Let dz denote the area measure on $\mathbb{D}$, and let the measure dv be given by the $\mathbf{SU}(1,1)$-invariant measure on $\mathbb{D}$ given by $dv(z) = dz/(1-|z|^2)^2$. The isometries are the Möbius-Blaschke transformations of $\mathbb{D}$ onto $\mathbb{D}$, given by

$$\varphi_{\theta,\alpha}(z) = e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z}\,, \alpha \in \mathbb{D}, \theta \in [0,2\pi)\,.$$

The Laplacian Δ on $\mathbb{D}$ is symmetric and given by

$$\Delta = (1-x^2-y^2)^{-2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Let $\langle z, b\rangle$ denote the minimal hyperbolic distance from the origin to the horocycle through z and a point $b \in \partial\mathbb{D}$. Eigenfunctions of Δ are of the form $e^{\beta\langle z,b\rangle}$. We have that

$$\Delta(e^{(i\lambda+1)\langle z,b\rangle}) = -(\lambda^2+1)e^{(i\lambda+1)\langle z,b\rangle}\,, \lambda \in \mathbb{C}\,,$$

and that the eigenfunctions of Δ are the functions

$$f(z) = \int_{\mathbb{T}} e^{(i\lambda+1)\langle z,b\rangle}\,d\mu(b)\,,$$

for $\lambda > 0$ and $b \in \mathbb{T}$, and where μ is an analytic functional on $\mathbb{T}$.

Let $\mathcal{E}(\mathbb{D})$ denote the set of C^∞ functions on $\mathbb{D}$, and let

$$\mathcal{E}_\lambda(\mathbb{D}) = \{f \in \mathcal{E}(\mathbb{D}) : \Delta(f) = -(\lambda^2+1)(f)\}\,.$$

For $\lambda \in \mathbb{C}$, let T_λ be the representation of $\mathbf{SU}(1,1)$ on the eigenspace of $\mathcal{E}_\lambda(\mathbb{D})$. Then T_λ is irreducible if and only if $i\lambda + 1 \notin 2\mathbb{Z}$.

Let $\mathbb{T} = \partial\mathbb{D}$. By identifying the eigenfunctions of Δ and determining the set of maps φ for which T_φ is irreducible, we can write down, for functions $f \in L^1(\mathbb{D}, dv)$, the *Fourier-Helgason transform*, which is defined as

$$\widehat{f}(\lambda, b) = \int_{\mathbb{D}} f(z) e^{(-i\lambda+1)\langle z,b\rangle}\, dv(z)$$

for $\lambda > 0$, $b \in \mathbb{T}$, and $dv(z) = dz/(1-|z|^2)^2$. The mapping $f \rightarrow \widehat{f}$ extends to an isometry $L^2(\mathbb{D}, dv) \rightarrow L^2(\mathbb{R}^+ \times \mathbb{T}, (2\pi)^{-1}\lambda \tanh(\lambda\pi/2) d\lambda\, db)$, i.e., the Plancherel formula becomes $\int_{\mathbb{D}} |f(z)|^2 \frac{dz}{(1-|z|^2)^2} =$

$$\frac{1}{2\pi}\int_{\mathbb{R}^+\times\mathbb{T}} |\widehat{f}(\lambda, b)|^2 \lambda \tanh(\lambda\pi/2) d\lambda\, db.$$

Here db denotes the normalized measure on the circle $\mathbb{T}$, such that $\int_{\mathbb{T}} db = 1$, and $d\lambda$ is Lebesgue measure on $\mathbb{R}$. The *Fourier-Helgason inversion formula* is

$$f(z) = \frac{1}{2\pi}\int_{\mathbb{R}^+}\int_{\mathbb{T}} \widehat{f}(\lambda, b) e^{(i\lambda+1)\langle z,b\rangle} \lambda \tanh(\lambda\pi/2)\, d\lambda\, db\,.$$

We also note that $\widehat{\Delta f}(\lambda, b) = -(\lambda^2+1)\widehat{f}(\lambda, b)$.

V. Sampling on a Hyperbolic Surface

The Uniformization Theorem demonstrates the importance of hyperbolic geometry. The only covering surface of Riemann sphere $\widetilde{\mathbb{C}}$ is itself, with the covering map being the identity. The plane $\mathbb{C}$ is the universal covering space of itself, the once punctured plane $\mathbb{C}\backslash\{z_0\}$ (with covering map $\exp(z-z_0)$), and all tori $\mathbb{C}/\Gamma$, where Γ is a parallelogram generated by $z \longmapsto z + n\gamma_1 + m\gamma_2$, $n, m \in \mathbb{Z}$ and γ_1, γ_2 are two fixed complex numbers linearly independent over $\mathbb{R}$. The universal covering space of every other Riemann surface is the hyperbolic disk $\mathbb{D}$. Let's refer to these surfaces as hyperbolic surfaces. Given a connected hyperbolic Riemann surface $\mathcal{S}$ and its universal covering space $\mathbb{D}$, $\mathcal{S}$ is isomorphic to $\mathbb{D}/\Gamma$, where the group Γ is isomorphic to the fundamental group of $\mathcal{S}$, $\pi_1(\mathcal{S})$. The corresponding universal covering is simply the quotient map which sends every point of $\mathbb{D}$ to its orbit under Γ.

A function $f \in L^2(\mathbb{D}, dv)$ is called *band-limited* if its Fourier-Helgason transform $\widehat{f}$ is supported inside a bounded subset $[0, \Omega]$ of $\mathbb{R}^+$. The collection of band-limited functions with band-limit inside a set $[0, \Omega]$ will be denoted $\mathbb{PW}_\Omega = \mathbb{PW}_\Omega(\mathbb{D})$. This shows that sampling is possible for band-limited functions. If $f \in \mathbb{PW}_\Omega(\mathbb{D})$, f satisfies the following Bernstein inequality – $\|\Delta^n f\| \leq (1 + |\Omega|^2)^{n/2}\|f\|$. We can find a natural number N such that for any sufficiently small r, there are points $x_j \in \mathbb{D}$ for which $B(x_j, r/4)$ are disjoint, $B(x_j, r/2)$ cover $\mathbb{D}$, and $1 \leq \sum_j \chi_{B(x_j,r)} \leq N$. Such a collection of $\{x_j\}$ will be called an (r, N)-lattice. Use this covering on the fundamental domain $\mathcal{D}$, which is a subregion of $\mathbb{D}$. At points of the covering intersecting the boundary of the universal cover, a refined covering is needed when $\rho = \operatorname{dist}(x_j, \partial\mathcal{D} < r/4)$. In this case, refine the covering in those covering elements with $B(x_j, \rho)$.

Let φ_j be smooth non-negative functions which are supported in $B(x_j, r/2)$ and satisfy that $\sum_j \varphi_j = 1_{\mathbb{D}}$ and define the operator

$$Tf(x) = P_\Omega\left(\sum_j f(x_j)\varphi_j(x)\right),$$

where P_Ω is the orthogonal projection from $L^2(\mathbb{D}, dv)$ onto $\mathbb{PW}_\Omega(\mathbb{D})$. By decreasing r (and thus choosing x_j closer) one can obtain the inequality $\|I - T\| < 1$, in which case T can be inverted by $T^{-1}f = \sum_{k=0}^{\infty}(I-T)^k f$. For given samples, we can calculate Tf and the Neumann series. This provides the recursion formula

$$f_{n+1} = f_n + Tf - Tf_n\,.$$

We have that $\lim_{n\to\infty} f_n = f$ with norm convergence. The rate of convergence is determined by the estimate $\|f_n - f\| \leq \|I - T\|^{n+1}\|f\|$.

Theorem V.1 (Irregular Sampling by Iteration)**.** *Let $\mathcal{S}$ be a hyperbolic Riemann surface with universal covering $\mathbb{D}$. Then there exists an (r, N)-lattice on $\mathcal{S}$ such that given $f \in \mathbb{PW}_\Omega = \mathbb{PW}_\Omega(\mathcal{S})$, f can be reconstructed from its samples on the lattice via the recursion formula*

$$f_{n+1} = f_n + Tf - Tf_n\,.$$

We have $f_{n+1} \rightarrow f$ as $n \rightarrow \infty$ in norm. The rate of convergence is $\|f_n - f\| \leq \|I - T\|^{n+1}\|f\|$.

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