Network Tomography in Hyperbolic Space

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***Abstract*—The paper addresses the problems of network analysis and network security, outlining a computationally feasible method of monitoring networks, and detecting (hyper)-active increase in subnetwork activity, such as one would see in viral or network attack activity. Additionally, it outlines a systematic method of detecting the source of activity, and if needed, isolate and/or shut-down subcomponents of the network.**

I. Network Tomography

A goal of network security is to keep traffic moving and keep it free of viruses. One use of network tomography allows the creation of a system that will detect viruses as early as possible and work simply on the geometry or structure of the network itself. We have developed a computationally efficient method to monitor traffic. We monitor specific connected subsets of arbitrary weighted graphs (regions of interest) from the input output map corresponding to paths that have crossed such regions and from this, to determine, for instance, congested areas or even anticipate areas that will get congested. This would allow a system manager to take measures to avoid the stoppage of traffic. Viruses are detected by observing a rapid increase in network activity. Results on network tomography give that the network monitoring can be associated to a problem similar to electrical impedance tomography (EIT) on graphs and indicate how it is also associated to the Radon transform on trees. From this, we develop a strategy to determine the weight ω for the case of general weighted graphs.

Given that we are working in graphs, we will need discrete versions of our tools, e.g., discrete Fourier and Radon transforms, and discrete Laplacians. We discuss weighted graphs, and how the weights change due to an increase of traffic. In this case, the network configuration remains the same. The weighted graph problem looks at a tree in a Riemannian manifold with assumptions that it is connected, or that we can get from one node to any other node in the network. We consider relatively simple regions of interest in a graph and suitable choices for the data of the ω-Neumann boundary value problem to produce a linear system of equations for the values of ω. The other problem looks at disruptions that occur when a hole appears in the network or an edge "ceases" to exist. In this case, the topology of the network has changed. We will not be addressing this problem, and refer the interested reader to the work of Robinson *et al.* [14]. For our development, we assume a stable network topology. Very deep work of Berenstein *et al.* [1], [2], [4], [5] on network tomography gives that the network monitoring can be associated to a problem similar to electrical impedance tomography (EIT) on graphs and indicate how it is also associated to the Radon transform on trees. From this work, we develop a strategy to determine the weight ω for the case of general weighted graphs. The natural tool to use in this context is the *Radon Transform*. We discuss it next, referring to [5], [11]. The Radon Transform $\mathcal{R}(f)$ of $f \in L^1(\mathbb{R}^2)$ is the mapping defined by the projection or line integral of f along all possible lines L, i.e., for $\xi \in \mathbb{T}$, $x \in \mathbb{R}^2$ and lines $p = \xi \cdot x$,

$$\mathcal{R}(f) = \int f(x)\delta(p - \xi \cdot x)\, dx\,.$$

An important computation gives $\mathcal{R}(e^{-\pi(x^2+y^2)}) = e^{-\pi p^2}$, i.e., the Gaussian. In higher dimensions, given a function $f \in L^1$, the Radon Transform of f is determined by integrating over each hyperplane in the space. Clearly, $\mathcal{R}(f)$ is linear, and is an even homogeneous function of degree -1, i.e., $\mathcal{R}(f)(sp, s\xi) = |s|^{-1}\mathcal{R}(f)(p,\xi)$. Letting Δ denote the Laplacian over the spatial variables,$\mathcal{R}(\Delta f) = \frac{\partial^2 \mathcal{R}(f)}{\partial p^2}$, where we note that the right-hand side is just the one dimensional Laplacian. If f also depends on time, we introduce the wave operator $\Box_n = \Delta - \frac{\partial^2}{\partial t^2}$, getting

$$\mathcal{R}(\Box_n f) = \Box_1 \mathcal{R}(f)\,.$$

Therefore, the Radon transform in n dimensions is localizable if and only if the wave equation is localizable. One can express this identity by saying that the Radon transform intertwines the wave operator $\Box_n$ with the wave operator $\Box_1$. It follows that the Radon transform cannot be localized in even dimensions. The n-dimensional Radon Transform $\mathcal{R}_n$ is related to the n dimensional Fourier Transform $\mathcal{F}_n$, by

$$\mathcal{R}_n(f) = \mathcal{F}_1^{-1}\mathcal{F}_n(f)\,,$$

the *Fourier slice* formula. This allows us to use Fourier methods in computations, and get relations of shifting, scaling, convolution, differentiation, and integration. Radon inversion is necessary to recover desired information about internal structure. The formula can be derived in an even and odd part, then unified analogously to the Fourier series. The unified inversion formula is $f = \mathcal{R}^\dagger \Upsilon_0 \mathcal{R}(f)$, where Υ_0 is the Helgason operator (see [11]).

We are interested in the discrete Radon transform on trees and its inversion formula. A graph G is a finite or countable collection V of vertices $v_j, j = 0, 1, ...$ and a collection E of edges $e_{jk} = (v_j, v_k)$, i.e., pairs of vertices. Given two vertices u and v, we say they are neighbors if (u, v) is an edge and denote this by $u \sim v$. A geodesic from one point to another is a collection of pairwise distinct vertices. Closed geodesics are also known as cycles, hence one can say that a tree is a connected graph without cycles. We say that a function f on the tree T is L^1 if $\sum |f(v)| < \infty$, where the sum is taken over all vertices in T. Given a geodesic Γ in T, we define the Radon transform on Γ by $\mathcal{R}f(\Gamma) = \sum_{v\in\Gamma} f(v)$. Given a node v, let $\eta(v)$ be the number of edges that contain v as an endpoint. This number is called the degree of the node. We will assume throughout that we always have $\eta(v) \geq 3$ to ensure that the Radon transform is injective. Under these conditions, the Radon transform on a tree is invertible.

We will derive inversion in the case where T is homogeneous and $\eta(v) \geq 3$, following the development in [5]. Given v, w two vertices in T that are connected by a path $(v = v_0, \dots, v_m = w)$, the *distance* between v and w is m, and we denote this by $[v, w] = m$. Let $v(n)$ be the number of vertices of T at a distance n from a fixed vertex of T. We have that $v(n) = 1$ if $n = 0$, $v(n) = (\eta(v) + 1)(\eta(v))^{n-1}$ if $n \geq 1$. For $f \in L^1(T)$, let $\mu_n f(v)$ be the average operator defined by

$$\mu_n f(v) = \frac{1}{v(n)} \sum_{[v,w]=n} f(w)\,,\; v \in T\,.$$

It can be shown that μ_n is a convolution with radial kernel $h_n(v, w) = 1/v(n)$ if $[v, w] = n$, $h_n(v, w) = 0$ if $[v, w] \neq n$. Let $\mathcal{R}^*$ be the dual Radon transform defined for $\Phi \in L^\infty(\Gamma)$ by

$$\mathcal{R}^*(\Phi)(v) = \int_{\Gamma_v} \Phi(\alpha) d\rho_v(\alpha)\,,$$

for each vertex $v \in T$, with respect to a suitable family $\{\rho_v : v \in T\}$ of measures on Γ_v, where Γ_v is the set of all of the geodesics containing the vertex v.

Let $\beta = q/(2(q + 1))$. In order to obtain the inversion of $\mathcal{R}$, we observe that $\mathcal{R}\mathcal{R}^*$ acts as a convolution operator given by the radial kernel $h = \beta h_0 + \sum_{n=1}^{\infty} 2\beta h_n$. The identity $\mathcal{R}^*\mathcal{R} = \beta\mu_0 + \sum_{n=1}^{\infty} 2\beta\mu_n$ holds in $L^1(T)$, where the series is absolutely convergent in the convolution operator norm on $L^2(T)$, thus providing a bounded extension of $\mathcal{R}\mathcal{R}^*$ to $L^2(T)$. The unique bounded extension to $L^2(T)$ of the operator $\mathcal{R}\mathcal{R}^*$ is invertible on $L^2(T)$, and its inverse is the operator

$$E = \frac{2(q + 1)^3}{q(q - 1)^2} \left[\mu_0 + \sum_{n=1}^{\infty} 2(-1)^n \mu_n\right],$$

which acts as the convolution operator with the radial kernel

$$\frac{2(q + 1)^3}{q(q - 1)^2} \left[h_0 + \sum_{n=1}^{\infty} 2(-1)^n h_n\right].$$

As above, this series converges absolutely in the convolution operator norm on $L^2(T)$; in particular, E is bounded. This gives us the following.

Theorem I.1. *The Radon transform* $\mathcal{R} : L^1(T) \longrightarrow L^\infty(\Gamma)$ *is inverted by*

$$E\mathcal{R}^*\mathcal{R}f = f\,.$$

In hyperbolic space, we define the Radon Transform of f by taking the integral over each geodesic in the space. Because of the hyperbolic distance, we have to assume that f is a continuous function with exponential decrease. Helgason has shown that the Radon Transform is a $1 - 1$ mapping on the space of continuous functions in hyperbolic space with exponential decrease (see [11], pp. 111–133). This then makes it the tool of choice when working in that geometry. Inversion again splits into even and odd dimensions (see [11], pp. 127–133).

Conventional tomography is associated to the Radon transform in Euclidean spaces. In comparison, electrical impedance tomography, or EIT, is associated to the Radon transform in the hyperbolic plane.

There exists two distinct variations to the problems that could arise in a network – increased traffic and changes in network topology. We discuss weighted graphs and how the weights change due to an increase of traffic. In this case, the network configuration remains the same. We consider relatively simple regions of interest in a graph and suitable choices for the data of the ω-Neumann boundary value problem to produce a linear system of equations for the values of ω. We assume a stable network topology.

To address the internet traffic problem, we must begin with the structure of the internet. Smale [17] gives us insight as to how one can use the tools of differential geometry to study circuits. Munzner [1], [3], [15] has proven that the internet has a hyperbolic structure. She showed that the natural geometric domain to use is the real hyperbolic space of dimensions two or three, the choice of the dimension being related to the density of the network.

The internet also has a weighted graph structure. In particular, it can be modeled as a weighted tree. Therefore, in order to deal with the network problems we are interested in, we need to develop a calculus. We define a graph G, vertices V, and edges E as before. For every edge, we can associate a non-negative number ω corresponding to the traffic along that edge. The value ω is the *weight* of the edge. A geodesic from one point to another is a collection of pairwise distinct vertices.

We can think of this in terms of electrical circuits. The value $\omega(p, q)$ is called the *conductance* of (p, q) and $1/\omega(p, q)$ the *resistance* of (p, q). Also ω is the *conductivity*. A function $u : V \to \mathbb{R}$ gives a current across each edge (p, q) by Ohm's law, the current from p to q, $I = \omega(p, q)(u(p) - u(q))$. The function u is called ω-harmonic if for each interior node p, $\sum_{q\in N(p)} \omega(p, q)(u(q) - u(p)) = 0$, where $N(p)$ is the set of nodes neighboring p. In other words, the sum of the currents flowing out of each interior node is zero, which is the discrete equivalent of Kirkhoff's law.

Let Φ a function defined at the boundary nodes. The network will acquire a unique ω-harmonic function u with $u(p) = \Phi(p)$ for each $p \in \partial G$, i.e., Φ induces u and u is the potential induced by Φ. Considering a conductor (p, q), the potential drop across this conductor is $Du(p, q) = u(p) - u(q)$. The potential function u determines a current $I_\Phi(p)$ through each boundary node p, by $I_\Phi(p) = \omega(p,q)(u(p) - u(q))$, q being an interior neighbor of p. The boundary function Φ is called the Dirichlet data and the boundary current I_Φ is called Neumann data. As in the continuous case, for each conductivity ω on E, the linear map $\Lambda_\omega \Phi$ from boundary functions to boundary functions, defined by $\Lambda_\omega \Phi = I_\Phi$, (the input-output map) is called the Neumann-to-Dirichlet map. The problem to consider is to recover the conductivity ω from $\Lambda_\omega \Phi$.

Electrical impedance tomography ideas can be effectively used in this context to determine the conductivity ω (weight) in the network from the knowledge of the Neumann-to-Dirichlet map associated to ω. They show that the conductivity ω can be uniquely determined and give an algorithm to compute ω. They also show the continuity of the inverse.

We do calculus on a weighted graph G as follows. We define the degree of a node x by $d_\omega x = \sum_{y \in V} \omega(x,y)$. To integrate a function $f : G \longrightarrow \mathbb{R}$, we compute

$$\int_V f d_\omega = \sum_{x \in V} f(x)\, d_\omega x\,.$$

The directional derivative $D_{\omega,y} f(x)$ and gradient $\nabla_\omega f(x)$ are given by, for $y \in V$,

$$D_{\omega,y} f(x) = [f(y) - f(x)]\sqrt{\frac{\omega(x,y)}{d_\omega x}}, \nabla_\omega f(x) = (D_{\omega,y} f(x)),$$

respectively. The weighted ω-Laplacian $\Delta_\omega f$ is given by

$$\Delta_\omega f = \sum_{y \in V} [f(y) - f(x)] \frac{\omega(x,y)}{d_\omega x}\,,\ x \in V\,.$$

If S is a subgraph of G, we define the boundary of S, ∂S, by

$$\partial S = \{z \in V : z \notin S \text{ and } z \sim y \text{ for } y \in S\}\,.$$

Also, let $\bar{S} = S \cup \partial S$. The outward normal derivative $\frac{\partial f}{\partial n_\omega}(z)$ at $z \in \partial S$ is given by

$$\frac{\partial f}{\partial n_\omega}(z) = \sum_{y \in S} [f(z) - f(y)] \frac{\omega(z,y)}{d'_\omega z}\,,$$

where $d'_\omega z = \sum_{y \in S} \omega(z,y)$.

In the case of planar finite weighted graphs, Berenstein and Chung (see [2], [3]) gave the uniqueness result, that is, any two weights ω_1 and ω_2 must coincide if the Neumann-to-Dirichlet map associated to ω_1 is equal to the Neumann-to-Dirichlet map associated to ω_2. The values of ω will increase or decrease depending on traffic. We can then compute the discrete Laplacian derivative Δ of a weighted subgraph (Δ_ω), getting the rate of traffic on the subnetwork (Neumann data). We can compute the weights on individual edges from the boundary value data (Dirichlet data). Below we will see from the following theorems how these conditions hold.

First, we must mention why the boundary will be studied. The *Minimum and Maximum Principle* points out why we focus on the boundary conditions within the network.

Theorem I.2. *Let* S *be a subgraph of a host graph* G *with a weight ω and $f : \overline{S} \to \mathbb{R}$ be a function.*

1) *If $\Delta_\omega f(x) \geq 0$, $x \in S$ and f has a maximum at a vertex in S, then f is constant.*
2) *If $\Delta_\omega f(x) \leq 0$, $x \in S$ and f has a minimum at a vertex in S, then f is constant.*
3) *If $\Delta_\omega f(x) = 0$, $x \in S$ and f has either a minimum or maximum in S, then f is constant.*
4) *If $\Delta_\omega f(x) = 0$, $x \in S$ and f is constant on the boundary ∂S, then f is constant.*

The Dirichlet boundary condition can be represented using the discrete analogue of the Laplacian. We let $\langle f, g \rangle_X = \sum_{x \in X} f(x) g(x)$.

Theorem I.3. *Let S be a subgraph of a host graph with a weight ω and $\sigma : \partial S \to \mathbb{R}$ be a given function. Then the unique solution f to the Dirichlet boundary value problem*

$$\begin{cases} \Delta_{\omega_j} f_j(x) = 0 & x \in S, \\ f|_{\partial S} = \sigma \end{cases}$$

can be represented as

$$f(x) = -\langle \gamma_\omega(x,\cdot), B_\sigma \rangle_{y \in S}, x \in S,$$

where

$$B_\sigma(y) = \sum \frac{\sigma(z)\omega(y,z)}{d_\omega y}, y \in S.$$

To see the other side, we look at the Neumann condition which uses integration by parts and Green's formula in this theorem.

Theorem I.4. *Let* S *be a subgraph of a host graph* G *with a weight ω and let $f : \overline{S} \to \mathbb{R}$, $g : S \to \mathbb{R}$, and $\psi : \partial S \to \mathbb{R}$ be functions with $\int_{\partial S} \psi = \int_S g$. Then the solution to the Neumann boundary value problem*

$$\begin{cases} \Delta_{\omega_j} f(x) = g(x) & x \in S, \\ \frac{\partial f}{\partial_\omega n}(z) = \psi(z) & z \in \partial S \end{cases}$$

is given by

$$f(x) = a_0 + \langle \Omega_\omega(x,\cdot), g \rangle_S - \langle \Gamma_\omega(x,\cdot), \psi \rangle_{\partial S},$$

where Ω_ω is the Green's function of Δ_ω on the graph $\overline{S}$ as a new host graph of S and a_0 is an arbitrary constant.

Theorem I.5 (Dirichlet's Principle)**.** *Assume that $f : \overline{S} \to \mathbb{R}$ is a solution to*

$$\begin{cases} -\Delta_\omega f = g & on S, \\ f|\partial S = \sigma. \end{cases}$$

Then

$$I_\omega[f] = min_{h \in A} I_\omega[h].$$

The key ingredient is the attempt to understand what happens in a network from "boundary measurements," that is, to determine whether all of the nodes and routers are working or not and also measure congestion in the links between nodes by means of introducing test packets in the "external" nodes, the routers. To understand the boundary measurements, we must look at the Neumann-to-Dirichlet map. We must decompose and understand how this map will allow us to reduce the network to a system of linear equations. With this method, we can compute the actual weights from the knowledge of the Dirichlet data for convenient choices of the input Neumann data in a way similar to that done for lattices. In the context of electrical networks, the map, N, takes currents on $\partial\Omega$ and gives voltages on $\partial\Omega$ and is represented by a Neumann matrix N by the Green function of this Neumann boundary value problem. The following is the Neumann-to-Dirichlet map.

Theorem I.6. *Let ω_1 and ω_2 be weights with $\omega_1 \le \omega_2$, and $f_1, f_2 : \overline{S} \to \mathbb{R}$ be functions satisfying for $j = 1, 2$,*

$$\begin{cases} \Delta_{\omega_j} f_j(x) = 0 & x \in S, \\ \frac{\partial f_j}{\partial n_{\omega_j}}(z) = \psi(z) & z \in \partial S \end{cases}$$

for any given function $\psi : \partial S \to \mathbb{R}$ with $\int_{\partial S} \psi = 0, j = 1, 2$. If it is assumed that

$$\begin{aligned} &(i) \quad \omega_1(z, y) = \omega_2(z, y) \text{ on } \partial S \times \partial S, \\ &(ii) \quad f_{1|\partial S} = f_{2|\partial S}, \end{aligned}$$

then

$$\begin{cases} f_1 \equiv f_2, \text{ on } \overline{S} \text{ and} \\ \omega_1(x, y) = \omega_2(x, y), \\ \text{whenever } f_1(x) \neq f_1(y), \text{ or } f_2(x) \neq f_2(y). \end{cases}$$

The discrete Radon transform is injective in this setting, and therefore invertible. If increased traffic is detected, we can use the inverse Radon transform to focus in on particular signals. Given that these computations are just matrix multiplications, the computations can be done in real time on suitable subnetworks.

Finally, a theorem by Berenstein and Chung give us uniqueness. We can solve for the information via the Neumann matrix N. We then use the Neumann-to-Dirichlet map to get the information as boundary values. Uniqueness carries through. Thus, each subnetwork is distinct and can be solved individually. This allows us to piece together the whole network as a collection of subnetworks, which it turn, can be solved uniquely as a set of linear equations. The key equation to solve is the following in the end. Set S be a network with boundary ∂S, let ω_1, ω_2 be weights on two paths in the network, and let f_1, f_2 be the amount of information on those paths, modeled as real valued functions. Then we wish to solve, for $j = 1, 2$

$$\begin{cases} \Delta_{\omega_j} f_j(x) = 0 & x \in S \\ \frac{\partial f_j}{\partial n_{\omega_j}}(z) = \psi(z) & z \in \partial S \\ \int_S f_j \, d\omega_j = K & . \end{cases}$$

Looking at the internet as modeled as a hyperbolic graph allows for the natural use of the Neumann-to-Dirichlet map, and thus the discrete Radon Transform. The inverse of the discrete Radon Transform $E\mathcal{R}^*\mathcal{R}$ completes the problem with its result giving the interior data.

We finish by discussing graph decompositions. Given an arbitrary network, we look to develop a scheme to decompose the structure so that it is a union of hyperbolic networks. We can pick any given node in a graph, and create the node's subgraph as an m-ary tree. The leaf nodes of each m-ary tree are further replaced by their own neighborhood m-ary trees. In this way, a K-level m-ary tree is recursively constructed for each vertex. The decomposition is not unique, which can be seen simply by altering the starting vertex of any subgraph. Using spectral graph sampling methods, it may be possible to replace the original graph by a union of much smaller graphs. The challenge is to ensure that the union of subgraphs is a good representation of the original graph. The current state of the art on this decomposition is the BC (Block Cut-vertex) tree, which represents the decomposition of a graph into biconnected components [12], [13]. They introduce two new families of proxy graph methods, BCP-W and BCP-E, tightly integrating graph sampling methods with the BC (Block Cut-vertex) tree, which represents the decomposition of a graph into biconnected components. The next tasks lie in computationally efficient graph decomposition, determining regions of interest, and decomposing into unions of hyperbolic trees.

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