Construction of Non-Uniform Parseval Wavelet Frames for $L^2(R)$ via UEP

Hari Krishan Malhotra and Lalit Kumar Vashisht
Department of Mathematics
University of Delhi
Delhi-110007, India.
Email: krishan.hari67@gmail.com, lalitkvashisht@gmail.com

***Abstract*—We study the construction of non-uniform Parseval wavelet frames for the Lebesgue space $L^2(R)$, where the related translation set is not necessary a group. The unitary extension principle (UEP) and generalized (or oblique) extension principle (OEP) for the construction of multi-generated non-uniform Parseval wavelet frames for $L^2(R)$ are discussed. Some examples are also given to illustrate our results.**

Index Terms– Frame, Non-uniform wavelet system, Unitary extension principle (UEP).

I. Introduction

A countable sequence $\{f_k\}_{k\in I}$ in a separable Hilbert space $\mathcal{H}$ is called a *Hilbert frame* for $\mathcal{H}$, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{k\in I} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \ f \in \mathcal{H}.$$

The scalars A and B are called frame bounds of the frame $\{f_k\}_{k\in I}$. The frame $\{f_k\}_{k\in I}$ is said to be *tight* if it is possible to choose $A = B$, and *Parseval frame* if $A = B = 1$. The operator $S : \mathcal{H} \to \mathcal{H}$, $S(f) = \sum_{k\in I}\langle f, f_k\rangle f_k$, $f \in \mathcal{H}$ is called the frame operator associated with the frame $\{f_k\}_{k\in I}$. The frame operator S is bounded, linear, self-adjoint, positive and invertible on $\mathcal{H}$. This gives the *reconstruction formula* for all $f \in \mathcal{H}$, $f = SS^{-1}f = \sum_{k\in I}\langle S^{-1}f, f_k\rangle f_k = \sum_{k\in I}\langle f, S^{-1}f_k\rangle f_k$, where the series converges unconditionally. The scalars $\langle f, S^{-1}f_k\rangle$ are called *frame coefficients*. Basic theory of frames and their applications in different directions can be found in the books of Casazza and Kutyniok [4], Christensen [6], Han, Kornelson, Larson, and Weber [18], the tutorials of Casazza [3], Heil and Walnut [19], and the memoir of Han and Larson [17].

One of the important types of frames that handle the computational aspects are tight frames. The tight frames are particularly desirable due to the fact that the reconstruction of a vector (signal) from tight frame coefficients is numerically stable, see [4, p. 26] for details. Furthermore, the dual of a tight frame has the same structure as that of the frame itself. In this paper, we discuss construction of non-uniform Parseval wavelet frames for the signal space $L^2(R)$, where the related translation set may not be a group. The unitary extension principle (UEP) and generalized (or oblique) extension principle for the construction of multi-generated non-uniform Parseval wavelet frames for $L^2(R)$ are discussed. Some examples are also given to illustrate our results.

A. Non-Uniform Wavelet System

Gabardo and Nashed [14] studied non-uniform wavelets, also see [15], where the related translation set may not be a group. In real life application all signals are not obtained from uniform shifts; so there is a natural question regarding analysis and decompositions of this types of signals by a stable mathematical tool. Gabardo and Nashed [14] and Gabardo and Yu [15] filled this gap by the concept of non-uniform multiresolution analysis. Let Z and R denote the set of all integers, and real numbers, respectively. Throughout the paper, N is a positive integer, r be an odd integer relatively prime to N such that $1 \leq r \leq 2N-1$ and Λ is given by

$$\Lambda = \left\{0, \frac{r}{N}\right\} + 2Z.$$

The discrete set Λ is not always a group. For $a, b \in R$, the following operators acting on $L^2(R)$, are called translation, modulation and dilation operators, respectively.

$$T_a : L^2(R) \to L^2(R), \quad T_a f(\gamma) = f(\gamma - a),$$
$$E_b : L^2(R) \to L^2(R), \quad E_b f(\gamma) = e^{2\pi i b\gamma} f(\gamma),$$
$$L : L^2(R) \to L^2(R), \quad Lf(\gamma) = \sqrt{2N} f(2N\gamma).$$

The j fold N-dilation, where $j \in Z$, is given by

$$L^j f(\gamma) = (2N)^{\frac{j}{2}} f((2N)^j\gamma).$$

Definition 1.1: Let $\{\psi_1, \psi_2, \ldots, \psi_n\} \subset L^2(R)$ be a finite set of non-zero functions. The family

$$\begin{aligned}&\{L^jT_\lambda\psi_\ell\}_{\substack{j\in Z,\lambda\in\Lambda\\ \ell=1,2,\ldots,n}}\\ &= \left\{(2N)^{\frac{j}{2}}\psi_1((2N)^j\gamma - \lambda)\right\}_{j\in Z,\lambda\in\Lambda} \bigcup \cdots\\ &\bigcup \left\{(2N)^{\frac{j}{2}}\psi_n((2N)^j\gamma - \lambda)\right\}_{j\in Z,\lambda\in\Lambda}\end{aligned}$$

is called a non-uniform wavelet system in $L^2(R)$.

Definition 1.2: A frame of the form $\{L^jT_\lambda\psi_\ell\}_{\substack{j\in Z,\lambda\in\Lambda\\ \ell=1,2,\ldots,n}}$ for $L^2(R)$ is called a non-uniform wavelet frame for $L^2(R)$. That is, if there exist finite positive constants a_o and b_o such that

$$a_o\|f\|^2 \leq \sum_{j\in Z}\sum_{\lambda\in\Lambda}\sum_{\ell=1}^{n} |\langle f, L^jT_\lambda\psi_\ell\rangle|^2 \leq b_o\|f\|^2, \ f \in L^2(R)$$

B. Motivation

The extension problem in frame theory has a long history, regarding extension of a Bessel sequence to a frame for the underlying space. Christensen, Kim, and Kim showed in [5] that the extension problem has a solution in the sense that "any Bessel sequence can be extended to a tight frame by adjoining a suitable family of vectors in a separable Hilbert space." Ron and Shen introduced unitary extension principle for construction of tight wavelet frames in the Lebesgue space $L^2(R^d)$, where conditions for the construction of multi-generated tight wavelet frames are based on a given refinable function. The unitary extension principle allows construction of tight wavelet frames with compact support, desired smoothness; and good approximation of functions. In the direction of construction of Parseval frames from non-uniform multiwavelets systems, we give a general setup and prove the unitary extension principle for construction of multi-generated non-uniform tight wavelet frames for $L^2(R)$. Some examples are also discussed.

II. UEP for Non-Uniform Wavelet Systems

In this section, we discuss the unitary extension principle (UEP) for non-uniform wavelet system. First we recall some basic notations and definitions to make the paper self-contained. The support of a function ψ is denoted by Supp ψ, and defined as Supp $\psi = closure\Big(\{x : \psi(x) \neq 0\}\Big)$. The symbol $\overline{z}$ denotes the complex conjugate of a complex number z. The conjugate transpose of a matrix H is denoted by H^*. The characteristic function of a set E is denoted by χ_E. The spaces $L^2(R)$ and $L^\infty(R)$ denote the equivalence classes of square-integrable functions and essentially bounded functions on R, respectively. Next, we recall the Parseval identity. Let $\{e_k\}_{k\in Z}$ be an orthonormal basis for a separable Hilbert space $\mathcal{H}$. Then,

$$\sum_{k\in Z} |\langle f, e_k\rangle|^2 = \|f\|^2, \ f \in \mathcal{H}.$$

The Fourier transform of a function f is denoted by $\mathcal{F}$ or $\hat{f}$, and defined as

$$\mathcal{F}f = \hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\gamma}dx.$$

For $j \in Z$ and $a \in R$, by direct calculation, we have the following properties.

1) $L^j : L^2(R) \to L^2(R)$ is unitary map.
2) $L^jT_a = T_{(2N)^{-j}a}L^j$.
3) $\mathcal{F}L^j = L^{-j}\mathcal{F}$.
4) $\mathcal{F}T_a = E_{-a}\mathcal{F}$.

Now we give a set of assumptions for the construction of non-uniform Parseval wavelet frames for the signal space $L^2(R)$. Let $\psi_0 \in L^2(R)$ be a non-zero function such that

1) $\hat{\psi}_0(2N\gamma) = H_0(\gamma)\hat{\psi}_0(\gamma), \ H_0(\gamma) \in L^\infty(R)$;
2) Supp $\hat{\psi}_0(\gamma) \subseteq [0, \frac{1}{4N}]$; and
3) $\lim\limits_{\gamma\to 0^+} \hat{\psi}_0(\gamma) = 1$.

Further, let $H_1, H_2, \ldots, H_n \in L^\infty(R)$, and define $\psi_1, \psi_2, \ldots, \psi_n \in L^2(R)$ such that

$$\widehat{\psi}_\ell(2N\gamma) = H_\ell(\gamma)\hat{\psi}_0(\gamma), \ \ell = 1, 2, \ldots, n.$$

Let $H(\gamma)$ be a $(n+1)\times 1$ matrix given by

$$H(\gamma) = \begin{pmatrix} H_0(\gamma) \\ H_1(\gamma) \\ \vdots \\ H_n(\gamma) \end{pmatrix} \tag{1}$$

We call the collection $\{\psi_\ell, H_\ell\}_{\ell=0}^n$, the *non-uniform general setup* (NGS).

The next two results are called UEP and OEP for non-uniform wavelet systems, we refer to the preprint [22] for technical details.

Theorem 2.1: [22] Let $\{\psi_\ell, H_\ell\}_{\ell=0}^n$ be a non-uniform general setup and $H(\gamma)^*H(\gamma) = 1$. Then, the non-uniform multiwavelets system $\{L^jT_\lambda\psi_\ell\}_{\substack{j\in Z,\lambda\in\Lambda \\ \ell=1,2,\ldots,n}}$ constitutes a Parseval frame for $L^2(R)$.

Theorem 2.2: [22] Let $\{\psi_\ell, H_\ell\}_{\ell=0}^n$ be a non-uniform general setup. Assume that there exist strictly positive function $\theta \in L^\infty(R)$ for which

$$(i) \ \lim_{\gamma\to 0^+} \theta(\gamma) = 1,$$

$$(ii) \ \theta(2N\gamma)|H_0(\gamma)|^2 + \sum_{\ell=1}^{n} |H_\ell(\gamma)|^2 = \theta(\gamma).$$

Then, $\{L^jT_\lambda\psi_\ell\}_{\substack{j\in Z,\lambda\in\Lambda \\ \ell=1,2,\cdots,n}}$ is a Parseval non-uniform wavelet frame for $L^2(R)$.

III. Applicative Examples

Example 3.1: Let $N = 2, r = 3$, and $\psi_0 \in L^2(R)$ be such that

$$\hat{\psi}_0(\gamma) = e^{i\gamma}\chi_{[0,\frac{1}{8}]}(\gamma).$$

Then

$$(i) \ \lim_{\gamma\to 0^+} \hat{\psi}_0(\gamma) = 1;$$

$$(ii) \ \textit{Supp } \hat{\psi}_0 \subseteq [0, \frac{1}{8}]; \ \textit{and}$$

$$(iii) \ \hat{\psi}_0(4\gamma) = e^{4i\gamma}\chi_{[0,\frac{1}{32}]}(\gamma)\chi_{[0,\frac{1}{8}]}(\gamma) = H_0(\gamma)\hat{\psi}_0(\gamma),$$

where $H_0(\gamma) = e^{3i\gamma}\chi_{[0,\frac{1}{32}]}(\gamma) \in L^\infty(R)$. Let $H_1 = \chi_{R\setminus[0,\frac{1}{32}]}$ and $\psi_1 \in L^2(R)$ be defined as

$$\hat{\psi}_1(2N\gamma) = \hat{\psi}_1(4\gamma) = H_1(\gamma)\hat{\psi}_0(\gamma).$$

Then, $\{\psi_\ell, H_\ell\}_{\ell=0}^1$ is a non-uniform general setup such that

$$H(\gamma)^*H(\gamma) = |H_0(\gamma)|^2 + |H_1(\gamma)|^2 = 1.$$

Hence, by Theorem 2.1, the family $\{L^jT_\lambda\psi_1\}_{j\in Z,\lambda\in\{0,\frac{3}{2}\}+2Z}$ is a non-uniform Parseval wavelet frame $L^2(R)$.

To conclude the paper, we illustrate Theorem 2.2 with the following example.

Example 3.2: Let $N=2$, $r=3$ and $\psi_0 \in L^2(R)$ be such that

$$\hat{\psi_0}(\gamma)=\frac{\sin(\gamma)}{\gamma}\chi_{[0,\frac{1}{8}]}(\gamma).$$

Then

$$(i)\ \lim_{\gamma\to 0^+}\hat{\psi_0}(\gamma)=1;$$

$$(ii)\ Supp\ \hat{\psi_0}(\gamma)\subseteq[0,\frac{1}{8}];\ and$$

$$(iii)\ \hat{\psi_0}(4\gamma)=\frac{4\sin(\gamma)\cos(\gamma)\cos(2\gamma)}{4\gamma}\chi_{[0,\frac{1}{32}]}(\gamma)\chi_{[0,\frac{1}{8}]}(\gamma)$$
$$=H_0(\gamma)\hat{\psi_0}(\gamma),$$

where $H_0(\gamma)=\cos(\gamma)\cos(2\gamma)\chi_{[0,\frac{1}{32}]}(\gamma)$.

Define

$$\begin{aligned}H_1(\gamma)\quad &=\cos(2\gamma)\sin(\gamma)\chi_{[0,\frac{1}{32}]}(\gamma),\\ H_2(\gamma)\quad &=\sin(2\gamma)\chi_{[0,\frac{1}{32}]}(\gamma),\ and\\ H_3(\gamma)\quad &=\sqrt{\chi_{]\frac{1}{32},\frac{1}{8}]}+2\chi_{R\setminus[0,\frac{1}{8}]}}.\end{aligned}$$

Let $\psi_1,\psi_2,\psi_3\in L^2(R)$ be such that

$$\hat{\psi_\ell}(4\gamma)=H_\ell(\gamma)\hat{\psi_0}(\gamma),\ \ell=1,2,3.$$

Then, the collection $\{\psi_\ell,H_\ell\}_{\ell=0}^3$ is a non-uniform general setup. Define strictly positive function

$$\theta(\gamma)=\chi_{[0,\frac{1}{8}]}+2\chi_{R\setminus[0,\frac{1}{8}]}\in L^\infty(R).$$

Then, $\lim_{\gamma\to 0^+}\theta(\gamma)=1$, and

$$\theta(4\gamma)|H_0(\gamma)|^2+|H_1(\gamma)|^2+|H_2(\gamma)|^2+|H_3(\gamma)|^2=\theta(\gamma).$$

Hence, by Theorem 2.2, the system $\{L^jT_\lambda\psi_\ell\}_{j\in Z,\ \lambda\in\{0,\frac{3}{2}\}+2Z,\ \ell=1,2,3}$ is a non-uniform Parseval wavelet frame for $L^2(R)$.

IV. Concluding Remark

The role of tight wavelet frames, in particular, Parseval frames in the study of perfect reconstruction and analysis of signals is well known. In Theorem 2.1 and Theorem 2.2, we present construction of Parseval frames of non-uniform wavelet systems, where the associated translation may not be an integer shift.

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