Conjugate Phase Retrieval in Paley-Wiener Space

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***Abstract*—We consider the problem of conjugate phase retrieval in Paley-Wiener space PW_π. The goal of conjugate phase retrieval is to recover a signal f from the magnitudes of linear measurements up to unknown phase factor and unknown conjugate, meaning $f(t)$ and $\overline{f(t)}$ are not necessarily distinguishable from the available data. We show that conjugate phase retrieval can be accomplished in PW_π by sampling only on the real line by using structured convolutions. We also show that conjugate phase retrieval can be accomplished in PW_π by sampling both f and f' only on the real line. Finally, we show that generically, conjugate phase retrieval can be accomplished by sampling at 3 times the Nyquist rate, whereas phase retrieval requires sampling at 4 times the Nyquist rate.**

I. Introduction

The phase retrieval problem can be stated as follows: can a signal f be reconstructed from the magnitudes of linear measurements of f? Naturally, f and αf cannot be distinguished by the magnitudes of linear measurements, where α is any scalar of magnitude 1. In general, one wishes to design a sampling scheme so that the magnitudes of linear measurements can distinguish all signals up to the ambiguity of this uniform phase factor. We consider in the present paper a weaker formulation of the problem: can a signal f be reconstructed from the magnitudes of linear measurements, up to the ambiguity of αf and $\alpha\overline{f}$? We refer to this as the *conjugate phase retrieval problem.*

Let us make precise our problem formulation here. The Paley-Wiener space PW_β consists of all $f \in L^2(\mathbb{R})$ such that $\hat{f}(\xi) = 0$ for a.e. $\xi \in \mathbb{R}\backslash[-\beta, \beta]$. Here, β is any positive number. Any such $f \in PW_\beta$ has an extension to an entire function on the complex plane. Moreover, if $f \in PW_\beta$, then the (entire) function $f^\sharp(z) := \overline{f(\overline{z})} \in PW_\beta$ as well. We define an equivalence relation on PW_β as follows: for $f, g \in PW_\beta$

$$f \sim g \text{ if } f = \alpha g, \text{ or } f = \alpha g^\sharp \text{ for some } |\alpha| = 1. \tag{1}$$

Our goal is to $a)$ design a sequence of linear functionals (measurements) $\phi_n : PW_\beta \to \mathbb{C}$ such that the mapping

$$\mathcal{A} : PW_\beta / \sim \to \ell^2(\mathbb{Z}) : f \mapsto (|\phi_n(f)|)_n \tag{2}$$

is one-to-one, and $b)$ reconstruct $[f]$ from $(|\phi_n(f)|)_n$, where $[f]$ denotes the equivalence class in $PW_\beta/\sim$ of $f \in PW_\beta$.

The phase retrieval problem originates in optics [1], [2], [3], [4]. Modern phase retrieval is often considered in the case of frames [5], [6], [7]. Conjugate phase retrieval for frames was introduced in [8]. Phase retrieval in the context of wavelets and other systems appear in [9], [10], [11], [12]. Phase retrieval in the Paley-Wiener space in particular is discussed in [13], [14]. In [13] considers the case of real phase retrieval in PW_π, meaning only real-valued signals f are sampled. The main result is that if one samples f at more than twice the Nyquist frequency, then $\pm f$ can be recovered from $(|f(t_n)|)_n$. We note here that the reconstruction of $\pm f$ given [13] involves reconstruction off of the real axis. Similarly, [14] considers the case of (complex) phase retrieval in PW_π by designing a sampling scheme that occurs off of the real axis. In particular, the sampling scheme as presented in [14] takes the form

$$\phi_n(f) = \sum_j c_{j,n} f(z_n - b_{j,n}) \tag{3}$$

for complex scalars $c_{j,n}, z_n, b_{j,n}$. Sampling schemes such as this are referred to as *structured modulations* in [14] because the authors there consider the reconstruction in the Fourier domain, where the shifts become modulations.

We will design sampling schemes for the conjugate phase retrieval problem in PW_π (our statements can be modified appropriately for PW_β). In Subsection II-B, our sampling scheme will take the form of structured convolutions. However, we will demonstrate that by solving the conjugate phase retrieval problem (which is weaker the the phase retrieval problem), we will be able to both sample and perform the reconstruction on the real axis. In Subsection II-E, we will show that the conjugate phase retrieval problem can be solved by sampling both f and f' (on the real axis as well) rather than with structured convolutions.

II. Conjugate Phase Retrieval

A. Preliminary Results

Our results are based on several elementary and known results. The first concerns the square of a signal $f \in PW_\beta$:

Lemma 1: If $f \in PW_\beta$, then:

1. $f' \in PW_\beta$;
2. $ff^\sharp \in PW_{2\beta}$;
3. $f'(f')^\sharp \in PW_{2\beta}$.

The following result is proven in [15]:

Proposition 1: Suppose $f, g \in PW_\beta$.

1. If $b < \beta/\pi$, and for all $x \in \mathbb{R}$, $|f(x)| = |g(x)|$ and $|f(x+b) - f(x)| = |g(x+b) - g(x)|$, then $f \sim g$.
2. If for all $x \in \mathbb{R}$, $|f(x)| = |g(x)|$ and $|f'(x)| = |g'(x)|$, then $f \sim g$.

In Proposition 1, $f \sim g$ is the equivalence relation given in Equation (1).

A sequence $\{t_n\}_n \subset \mathbb{R}$ is a *set of sampling* for PW_β provided that there exist constants $0 < A, B$ such that

$$A\|f\|^2 \leqslant \sum_n |f(t_n)|^2 \leqslant B\|f\|^2$$

holds for all $f \in PW_\beta$.

Theorem 1: Suppose $\{t_n\} \subset \mathbb{R}$ is a set of sampling for $PW_{2\beta}$. Then the mapping

$$\begin{aligned}\mathcal{A} : PW_\beta/\sim &\to \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})\\ : f &\mapsto (|f(t_n)|, |f(t_n + b) - f(t_n)|)_n\end{aligned}$$

is one-to-one whenever $b < \frac{\beta}{\pi}$.

Similarly, the mapping

$$\begin{aligned}\widetilde{\mathcal{A}} : PW_\beta/\sim &\to \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})\\ : f &\mapsto (|f(t_n)|, |f'(t_n)|)_n\end{aligned}$$

is one-to-one.

The proof follows from the fact that $f(t)f^\sharp(t)$ and $(f(t+b) - f(t))(f(t+b) - f(t))^\sharp$ can be reconstructed from the sequence of samples $(|f(t_n)|^2)_n$ $(|f(t_n + b) - f(t_n)|^2)_n$, respectively, which we note can be done in a stable way from the hypotheses.

Theorem 2: The range $\mathcal{R}(\mathcal{A})$ is closed. The inverse $\mathcal{A}^{-1}$ is continuous from $\mathcal{R}(\mathcal{A})$ to $PW_\beta/\sim$. The same results hold for $\widetilde{\mathcal{A}}$.

The proof of this is an adaptation of a similar result found in [10]. The authors of [10] note that in their numerical experiments, the reconstruction is not stable. It is proven in [11] that $\mathcal{A}^{-1}$ is not Lipschitz continuous and thus the reconstruction cannot be stable; the reason for the lack of stability is because the space PW_β is infinite-dimensional, and not because of a defect in any sampling scheme.

B. Conjugate Phase Retrieval Using Structured Convolutions

We will design a sampling scheme to solve the conjugate phase retrieval problem in PW_β in a manner similar to the scheme in Equation (3). To do so, we consider the conjugate phase retrieval problem in finite dimensions.

Definition 1: The vectors $\{\vec{v}_1, \dots, \vec{v}_n\} \subset \mathbb{C}^K$ do *conjugate phase retrieval* if for every $\vec{x}, \vec{y} \in \mathbb{C}^K$,

$$\begin{aligned}&|\langle \vec{x}, \vec{v}_j\rangle| = |\langle \vec{y}, \vec{v}_j\rangle| \text{ for } j = 1, \dots, n,\\ &\qquad \Rightarrow \vec{x} = e^{i\theta}\vec{y} \text{ or } \vec{x} = e^{i\theta}\overline{\vec{y}} \text{ for some } \theta \in \mathbb{R}.\end{aligned}$$

If we write the vectors $\vec{v}_j$ as column vectors, we will say that the matrix $A = \begin{bmatrix}\vec{v}_1 & \dots & \vec{v}_n\end{bmatrix}$ does conjugate phase retrieval when the columns of A do conjugate phase retrieval.

For this we require a result from [8] concerning conjugate phase retrieval in $\mathbb{C}^2$ and $\mathbb{C}^3$:

Proposition 2: If $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^2$ is written as

$$\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2\end{bmatrix}$$

then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ does conjugate phase retrieval in $\mathbb{C}^2$ if and only if

$$\det \begin{bmatrix} a_1^2 & 2a_1a_2 & a_2^2 \\ b_1^2 & 2b_1b_2 & b_2^2 \\ c_1^2 & 2c_1c_2 & c_2^2 \end{bmatrix} \neq 0. \tag{4}$$

Likewise, if $\vec{v}_1, \dots, \vec{v}_6 \in \mathbb{R}^3$ is written as

$$\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_6\end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \end{bmatrix}$$

then $\vec{v}_1, \dots \vec{v}_6$ does conjugate phase retrieval in $\mathbb{C}^3$ if and only if

$$\det \begin{bmatrix} a_1^2 & a_2^2 & a_3^2 & 2a_1a_2 & 2a_1a_3 & 2a_2a_3 \\ b_1^2 & b_2^2 & b_3^2 & 2b_1b_2 & 2b_1b_3 & 2b_2b_3 \\ c_1^2 & c_2^2 & c_3^2 & 2c_1c_2 & 2c_1c_3 & 2c_2c_3 \\ d_1^2 & d_2^2 & d_3^2 & 2d_1d_2 & 2d_1d_3 & 2d_2d_3 \\ e_1^2 & e_2^2 & e_3^2 & 2e_1e_2 & 2e_1e_3 & 2e_2e_3 \\ f_1^2 & f_2^2 & f_3^2 & 2f_1f_2 & 2f_1f_3 & 2f_2f_3 \end{bmatrix} \neq 0. \tag{5}$$

Theorem 3: Let $A = (a_{km})$ be a $K \times M$ matrix which does conjugate phase retrieval on $\mathbb{C}^K$. Let $\{b_j\}_{j=0}^{K-1} \subset \mathbb{R}$ be such that the group $\mathbb{Z}(\{b_0, b_1, \dots, b_{K-1}\})$ has finite upper Beurling density and lower Beurling density greater than one. Suppose $\{t_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ is a set of sampling for the space $PW_{2\pi}$. Then the following sampling scheme does conjugate phase retrieval on PW_π:

$$\{|\alpha_m * f(t_n)| : m = 0, 1, \dots, M-1;\ n \in \mathbb{Z}\}$$

where

$$\alpha_m * f = \sum_{k=0}^{K-1} \overline{a_{km}} f(\cdot - b_k). \tag{6}$$

Proof: Suppose $f, g \in PW_\pi$ is such that

$$|\alpha_m * f(t_n)| = |\alpha_m * g(t_n)|, \text{ for } m = 0, 1, \dots, M-1;\ n \in \mathbb{Z}. \tag{7}$$

Since $\{t_n\}$ is a set of sampling for $PW_{2\pi}$ and $|\alpha * f|^2, |\alpha * g|^2 \in PW_{2\pi}$, we have that

$$|\alpha_m * f(x)| = |\alpha_m * g(x)|, \text{ for all } x \in \mathbb{R}.$$

Note that the value $|\alpha_m * f(x)|$ is the magnitude of the inner product of the m-th column of A with the column vector $(f(x - b_0) \dots f(x - b_{K-1}))^T$. Since the matrix A does conjugate phase retrieval in $\mathbb{C}^K$, for all $x \in \mathbb{R}$ we have that either

$$\begin{pmatrix} f(x-b_0) \\ f(x-b_1) \\ \vdots \\ f(x-b_{K-1}) \end{pmatrix} = \lambda_1(x) \begin{pmatrix} g(x-b_0) \\ g(x-b_1) \\ \vdots \\ g(x-b_{K-1}) \end{pmatrix} \tag{8}$$

or

$$\begin{pmatrix} f(x-b_0) \\ f(x-b_1) \\ \vdots \\ f(x-b_{K-1}) \end{pmatrix} = \lambda_2(x) \begin{pmatrix} \overline{g(x-b_0)} \\ \overline{g(x-b_1)} \\ \vdots \\ \overline{g(x-b_{K-1})} \end{pmatrix} \tag{9}$$

for some $\lambda_j(x) \in \mathbb{C}$ with $|\lambda_j(x)| = 1$.

For every $k = 1, \dots, K-1$ and every x such that Equation (8) holds, we have that

$$\begin{aligned}|f(x-b_k) - f(x)| &= |\lambda_1(x)g(x-b_k) - \lambda_1(x)g(x)| \\ &= |g(x-b_k) - g(x)|.\end{aligned}$$

Similarly, for x such that Equation (9) holds, we have that

$$\begin{aligned}|f(x-b_k) - f(x)| &= |\lambda_2(x)\overline{g(x-b_k)} - \lambda_2(x)\overline{g(x)}| \\ &= |g(x-b_k) - g(x)|.\end{aligned}$$

Therefore, we have $|f(x)| = |g(x)|$ and $|f(x-b_k) - f(x)| = |g(x-b_k) - g(x)|$ both hold for all $k = 1, \dots, K-1$ and all $x \in \mathbb{R}$. By the proof of Proposition 1, we obtain that there exists a meromorphic function W such that either $f = Wg$ or $f = Wg^\sharp$. Moreover, W is periodic with period b_k for every $k = 1, \dots, K-1$, so it is periodic by the group $\mathbb{Z}(\{b_0, b_1, \dots, b_{K-1}\})$. Again by the proof of Proposition 1, since the group has density at least 1, W must be constant. ■

Theorem 4: Let $A = (a_{km})$ be a $K \times M$ matrix which does phase retrieval on $\mathbb{C}^K$. Let $\{b_j\}_{j=0}^{K-1} \subset \mathbb{R}$ be such that the group $\mathbb{Z}(\{b_0, b_1, \dots, b_{K-1}\})$ has finite upper Beurling density and lower Beurling density greater than one. Suppose $\{t_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ is a set of sampling for the space $PW_{2\pi}$. Then the following sampling scheme does phase retrieval on PW_π:

$$\{|\alpha_m * f(t_n)| : m = 0, 1, \dots, M-1;\ n \in \mathbb{Z}\}$$

where $\alpha_m * f$ are as in Equation (6).

The proof is identical to the proof of Theorem 3. We note, however, that the condition on the coefficient matrix A for the structured convolutions in Theorem 4 is much more restrictive than in Theorem 3. In fact, we shall see in Subsection II-D an illustration of this distinction. We also note that this generalizes the results in [14].

C. A reconstruction method:

The proof of Theorem 3 suggests a reconstruction method. We wish to reconstruct $f \in PW_\pi$ from the samples

$$\{|\alpha_m * f(t_n)| : m = 0, 1, \dots, M-1;\ n \in \mathbb{Z}\} \tag{10}$$

where $\{t_n\}$ and α_m satisfy the hypotheses of Theorem 3. However, we must structure the convolutions so that the $b_j = j/B$, for some $B > 1$, and $K \geqslant 3$, though $K = 3$ suffices.

We need the following elementary lemmas.

Lemma 2: The set of all $\beta \in \mathbb{R}$ such that

$$f(\frac{n}{B} - b_j - \beta) = 0,$$

for some $j = 0, \dots, K-1$ and for some $n \in \mathbb{Z}$ is countable.

Lemma 3: Suppose g is an entire function. For fixed $\{b_0, \dots, b_{K-1}\} \subset \mathbb{R}$, the set of $x \in \mathbb{R}$ for which the vectors

$$\begin{pmatrix} g(x-b_0) \\ g(x-b_1) \\ \vdots \\ g(x-b_{K-1}) \end{pmatrix} \text{ and } \begin{pmatrix} \overline{g(x-b_0)} \\ \overline{g(x-b_1)} \\ \vdots \\ \overline{g(x-b_{K-1})} \end{pmatrix}$$

are colinear is either all of $\mathbb{R}$ or at most countable.

Reconstruction Algorithm 1:

1. From the samples in Equation (10), reconstruct the functions
$$|\alpha_m * f(x)|^2, \quad m = 0, 1, \dots, M-1,$$
using the Shannon sampling theorem.
2. Choose β at random†; by Lemma 2, with probability 1,
$$f(\frac{n}{B} - b_j - \beta) \neq 0 \text{ for all } j = 0, \dots, K-1,\ n \in \mathbb{Z}.$$
3. Calculate the following samples using Step 1:
$$|\alpha_m * f(\frac{n}{B} - \beta)|^2, \quad m = 0, 1, \dots, M-1,\ n \in \mathbb{Z}.$$
4. Use the fact that the matrix A does conjugate phase retrieval to calculate for each $n \in \mathbb{Z}$ the vector
$$\vec{F}_n := \lambda(\frac{n}{B} - \beta) \begin{pmatrix} f(\frac{n}{B} - b_0 - \beta) \\ f(\frac{n}{B} - b_1 - \beta) \\ \vdots \\ f(\frac{n}{B} - b_{K-1} - \beta) \end{pmatrix} \tag{11}$$
up to the unknown phase $\lambda(\frac{n}{B} - \beta)$ and unknown conjugation.
5. For adjacent vectors $\vec{F}_n$ and $\vec{F}_{n+1}$, choose the conjugations and phase factors so that the entries that appear in both vectors agree. This can be done, since by Lemma 3, the choice of β makes these choices unique (with probability 1).
6. We now obtain the samples
$$\{\lambda f(\frac{n}{B} - \beta) : n \in \mathbb{Z}\} \text{ or } \{\overline{\lambda f(\frac{n}{B} - \beta)} : n \in \mathbb{Z}\}$$
up to unknown unimodular scalar λ, depending on whether our choice for conjugation was correct. Using these samples, we reconstruct $\lambda f(x - \beta)$ (if our choice of conjugation was correct) or $\lambda f^\sharp(x - \beta)$ (if our choice of conjugation was incorrect).

† By random we mean with respect to any continuous probability distribution on $\mathbb{R}$ or $[0, 1]$. This holds because there are at most countably many β that fail to have the required property, and for any continuous probability distribution, a countable set has probability 0.

D. An Example

We demonstrate here a sampling scheme using simple structured convolutions and the corresponding reconstruction as outlined in Reconstruction Algorithm 1 to do conjugate phase retrieval in PW_π. For this sampling scheme we are considering the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

which does not do phase retrieval, but does conjugate phase retrieval by Proposition 2. Note that $M = 3$ and $K = 6$. We choose a $0 < b \leqslant 1/2$, and choose $b_k = kb$ for $k = 0, 1, 2$.

For the 6 column vectors of A we sample $|\alpha_m * f(bn)|$, $n \in \mathbb{Z}$. However, the samples for columns 2 and 3 are repetitions of column 1, and column 6 is a repetition of column 4, and so we actually only need to sample the structured convolutions for columns 1, 4, and 5:

$\{|f(bn)|\}$, $\{|f(bn+b)-f(bn)|\}$, and $\{|f(bn+b)-f(bn-b)|\}$.

In Step 5 of Reconstruction Algorithm 1, we reconstruct

$$\vec{F}_n = \lambda(bn) \begin{pmatrix} f(bn+b-\beta) \\ f(bn-\beta) \\ f(bn-b-\beta) \end{pmatrix}$$

up to unknown phase $\lambda(bn)$ and conjugate. Note that two successive vectors have two sample points in common. The choice of β ensures that those two sample points are nonzero, and the two-dimensional vector is not colinear with its conjugate. Therefore, the phase and the conjugate for vector $\vec{F}_{n+1}$ can be determined from the phase and conjugate for vector $\vec{F}_n$.

We note here that our sampling scheme requires sampling 3 functions at more than twice the Nyquist rate, and thus our oversampling factor is at least 6. We can reduce this down to oversampling by a factor of 3 by incorporating our choice of β into the sampling scheme:

1. Choose β at random.
2. Sample $|\alpha_m * f(n-\beta)|$ for $m = 1, 4, 5$ and $n \in \mathbb{Z}$.
3. For each n, use the samples in Step 2 to reconstruct the vector
$$\vec{F}_n = \lambda(n) \begin{pmatrix} f(n+1-\beta) \\ f(n-\beta) \\ f(n-1-\beta) \end{pmatrix}$$
up to unknown phase $\lambda(n)$ and unknown conjugation.
4. Choose the phase and conjugation for $\vec{F}_{n+1}$ from the choice of phase and conjugation for $\vec{F}_n$, since they have 2 entries that coincide.

We note that the choice in Step 4 is unique, since as before, the choice of β yields that all of the entries of $\vec{F}_n$ are nonzero and the vector of overlapping entries is not colinear with its conjugate. However, this algorithm will only work on *generic* signals $f \in PW_\pi$, but not all signals. The set of signals for which this algorithm fails is a meager set in PW_π by the Baire Category Theorem.

It is known that in $\mathbb{C}^K$, a frame must have at least $4K-4$ vectors in order to do phase retrieval [5]. No such bound is known for conjugate phase retrieval, but note that our sampling scheme above suggests that it should be on the order of $3K$.

E. Conjugate Phase Retrieval using Derivatives

In analogy to structured convolutions, conjugate phase retrieval is possible by sampling the derivative of the signal.

Lemma 4: Suppose f and g are entire functions with the property that $ff^\sharp = gg^\sharp$ and $f'f'^\sharp = g'g'^\sharp$. Then there exists a unimodular scalar λ such that either $f = \lambda g$ or $f = \lambda g^\sharp$.

Theorem 5: Suppose $\{t_n\}$ is a set of sampling for $PW_{2\pi}$. Then the mapping

$$\begin{aligned} \widetilde{\mathcal{A}} &: PW_\pi / \sim \longrightarrow \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}) \\ &: f \mapsto (|f(t_n)|, |f'(t_n)|)_n \end{aligned}$$

is one-to-one.

We write

$$f(t) = r(t)e^{i\theta(t)} \quad t \in \mathbb{R},\ r(t) \geqslant 0,\ \theta(t) \in \mathbb{R}. \tag{12}$$

The functions r, θ are differentiable a.e. Theorem 5 and Lemma 4 provide a theoretical (but not a feasible numerical) reconstruction algorithm as follows.

Reconstruction Algorithm 2: Given the phaseless samples $\{|f(t_n)|, |f'(t_n)|\}$, proceed as follows:

1. reconstruct $ff^\sharp$ and $f'f'^\sharp$ in $PW_{2\pi}$;
2. reconstruct $r = \sqrt{ff^\sharp}$;
3. reconstruct
$$(\theta')^2 = \frac{f'f'^\sharp}{ff^\sharp} - \frac{[(ff^\sharp)']^2}{4(ff^\sharp)^2}$$
on some interval I;
4. choose a square-root of $(\theta')^2$ and integrate;
5. use $f = re^{i\theta}$ on I to expand f as a power series.

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