# Rearranged Fourier Series and Generalizations to Non-Commutative Groups

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***Abstract*—It is well-known that the Fourier series of continuous functions on the torus are not always uniformly convergent. However, P. L. Ulyanov proposed a problem: can we permute the Fourier series of each individual continuous function in such a way as to guarantee uniform convergence of the rearranged Fourier series? This problem remains open, but nonetheless a rather strong partial result was proved by S. G. Révész which states that for every continuous function there exists a subsequence of rearranged partial Fourier sums converging to the function uniformly.**

**We give several new equivalences to Ulyanov's problem in terms of the convergence of the rearranged Fourier series in the strong and weak operator topologies on the space of bounded operators on $L_2(\mathbb{T})$. This new approach gives rise to several new problems related to rearrangement of Fourier series. We also consider Ulyanov's problem and Révész's theorem for reduced $C^*$–algebras on discrete countable groups.**

## I. Introduction

For a function $f \in C(\mathbb{T})$, the formal series

$$S[f](t) := \sum_{n\in\mathbb{Z}} c_n e^{2\pi int}, \quad t \in \mathbb{T} \tag{1}$$

is called its Fourier series where $c_n$ is the $n$–th Fourier coefficient of $f$ defined by

$$c_n := \int_{\mathbb{T}} f(t)e^{-2\pi int}dt.$$

Even though the series in (1) are convergent in $L_2$ and pointwise almost everywhere (due to Carleson's theorem), they are not necessarily convergent in the inherent topology of $C(\mathbb{T})$ arising from the uniform norm. For example, a result of Kahane and Katznelson [7] shows that for any Lebesgue null set, there exists a continuous function whose Fourier series diverges on that set.

Appealing to intuition from other classical results on conditionally converging series, e.g. Riemann's rearrangement theorem, one might naturally suspect if reordering the terms in the Fourier series gives rise to a sequence of partial sums with better behavior. To set some notation, given a bijection $\sigma : \mathbb{Z} \to \mathbb{Z}$, define the rearranged partial sum of the Fourier series of $f \in L_1(\mathbb{T})$ by

$$S_{\sigma,N}[f](t) := \sum_{|n|\leq N} c_{\sigma(n)} e^{2\pi i\sigma(n)t}, \quad t \in \mathbb{T}.$$

In 1964, P. L. Ulyanov proposed the following problem [14].

**Problem 1** (Ulyanov)**.** *Given any $f \in C(\mathbb{T})$, does there exists a permutation $\sigma : \mathbb{Z} \to \mathbb{Z}$ such that the rearranged partial sums $S_{\sigma,N}[f]$ converge to $f$ uniformly as $N \to \infty$?*

Surprisingly, there are some rather strong partial results toward this problem. The next theorem is due to S. G. Révész [12] and serves as a starting point for our investigation.

**Theorem I.1** (Révész)**.** *For every $f \in C(\mathbb{T})$, there exists a permutation $\sigma : \mathbb{Z} \to \mathbb{Z}$ and a bijection $\{N_k\} \subset \mathbb{N}$ such that $S_{\sigma,N_k}[f]$ converges to $f$ uniformly as $k \to \infty$.*

A stronger result was later proved by S. Konyagin [8], [9]:

**Theorem I.2** (Konyagin)**.** *If $f \in C(\mathbb{T})$ has modulus of continuity satisfying $\omega(f, \delta) = o(1/\log\log(1/\delta))$, $\delta \to 0^+$, then there exists a permutation $\sigma : \mathbb{Z} \to \mathbb{Z}$ such that $S_{\sigma,N}[f]$ converges to $f$ uniformly as $N \to \infty$.*

Note that this decay condition on the modulus of continuity is mild. The classic Dini–Lipschitz criterion gives that if $\omega(f, \delta) = o(1/\log(1/\delta)$, $\delta \to 0^+$, then $S_N[f]$ converges uniformly to $f$, but Theorem I.2 shows that for much slower decay, this is true up to rearrangement.

Another facet of the conjecture has been considered by McNeal and Zeytuncu [10], wherein they demonstrate the existence of a rearrangement $\sigma$ which preserves pointwise convergence of partial sums of Fourier series (i.e. if $S_N[f](t)$ converges, then so does $S_{\sigma,N}[f](t)$). Additionally, they show that convergence of rearranged partial sums under such a permutation need not imply that the non-rearranged partial sums converge.

In Section II, we provide an operator theoretic equivalence to Ulyanov's problem. This approach naturally leads to other operator theoretic questions which are presented as theorems or problems.

In Section III we consider the analogue of Fourier series indexed by a discrete countable group and consider analoguous problems in that setting.

This paper may be considered an announcement of the results of [6], and thus proofs are typically omitted; however, the interested reader may find them there for the classical Fourier series case, while results in the non-commutative case will be the purview of future work.

## II. Classical Fourier series

**Definition 1.** For $f \in C(\mathbb{T})$, we say that $f_N \in C(\mathbb{T})$ converges to $f \in C(\mathbb{T})$ in the strong operator topology (SOT) if

$$f_N \cdot g \to f \cdot g \text{ in } L_2(\mathbb{T}) \text{ for every } g \in L_2(\mathbb{T}). \quad (2)$$

And we say that the convergence is in the weak operator topology (WOT) if

$$\int_{\mathbb{T}} f_N(t)g(t)dt \to \int_{\mathbb{T}} f(t)g(t)dt \text{ for every } g \in L_1(\mathbb{T}). \quad (3)$$

Note here that we are assigning to any continuous function $f$ a multiplication operator $M_f : L_2 \to L_2$ via multiplication, i.e. $M_f g = fg$ almost everywhere. Thus Definition 1 simply mean that $M_{f_N} \to M_f$ in SOT or WOT in the classic way, but to make terminology and notation easy, we define SOT or WOT convergence of continuous functions in this way.

The next theorem states that uniform convergence in Ulyanov's problem can be replaced with convergence in SOT or WOT.

**Theorem II.1.** *The following statements are equivalent:*

*(i) Ulyanov's problem has a positive answer, i.e. for every $f \in C(\mathbb{T})$, there is a bijection $\sigma\colon \mathbb{Z} \to \mathbb{Z}$ so that $S_{\sigma,N}[f] \to f$ uniformly,*

*(ii) For every $f \in C(\mathbb{T})$, there is a bijection $\sigma\colon \mathbb{Z} \to \mathbb{Z}$ so that $S_{\sigma,N}[f] \to f$ in the strong operator topology,*

*(iii) For every $f \in C(\mathbb{T})$, there is a bijection $\sigma\colon \mathbb{Z} \to \mathbb{Z}$ so that $S_{\sigma,N}[f] \to f$ in the weak operator topology.*

Note that for any fixed $f \in C(\mathbb{T})$, uniform convergence implies SOT convergence implies WOT convergence, but for arbitrary functions the converses of each of these implications is untrue. Thus this theorem allows us to examine seemingly weaker versions of Ulyanov's problem; in particular, by considering the conditions (2) and (3) on proper subspaces of $L_2(\mathbb{T})$ and $L_1(\mathbb{T})$, respectively. Along these lines, we propose the following relaxation of Ulyanov's problem.

**Problem 2.** *Is it true that for every $f \in C(\mathbb{T})$, and every finite dimensional subspace $V \subset L_2(\mathbb{T})$ (resp. $L_1(\mathbb{T})$) there exists a bijection $\sigma\colon \mathbb{Z} \to \mathbb{Z}$ so that* (2) *(resp.* (3)*) holds for all $g \in V$.*

With this notion of convergence in mind, it is interesting to consider what functions converge unconditionally in SOT and WOT. The following is a complete characterization of such functions.

**Theorem II.2.** *Let $\mathcal{W}$ be the set of functions $f \in C(\mathbb{T})$ whose Fourier series converge unconditionally in SOT. Then $\mathcal{W} = \mathcal{A}(\mathbb{T})$, the Wiener algebra of functions with absolutely convergent Fourier series. The statement also holds true if SOT is replaced with WOT.*

The next theorem can be regarded as the dual version of Theorem II.2, in which we characterize the largest subspace of $L_2$ for which the Fourier series of all continuous functions are unconditionally convergent in SOT.

**Theorem II.3.** *Let $\mathcal{V}$ be the set of functions $g \in L_2(\mathbb{T})$ so that $\|S_{\sigma,N}[f]g - fg\|_{L_2} \to 0$ for all $f \in C(\mathbb{T})$, $g \in \mathcal{V}$ and every bijection $\sigma\colon \mathbb{Z} \to \mathbb{Z}$. Then $\mathcal{V} = L_\infty(\mathbb{T})$.*

Interestingly, the statement of this theorem turns out to be equivalent to the following statement: there exists an absolute constant $c > 0$ such that for any measurable set $E \subseteq \mathbb{T}$ with $|E| > 0$ there exists a function $f \in C(\mathbb{T})$ with $\|f\|_\infty \leq 1$, and there exists a sequence $\mathcal{E} = \{\varepsilon_n\} \in \{-1, 1\}^{\mathbb{Z}}$ for which

$$\|T_{\mathcal{E}}[f]\mathbb{1}_E\|_2 \geq c$$

where

$$T_{\mathcal{E},N}[f](t) := \sum_{|n|\leq N} \varepsilon_n c_n e^{2\pi i n t},$$

and $T_{\mathcal{E}}[f]$ is the $L_2$ limit of $T_{\mathcal{E},N}[f]$, which is guaranteed to exist since $\{c_n\} \in \ell_2$. In [6], we show that $c \geq \frac{2}{3\sqrt{3\pi}}$ by a construction involving flat polynomials on the torus; however we conjecture that the supremum of all such $c$ for which the above statement holds is in fact equal to 1 (note that $c \leq 1$ by basic considerations).

The following is the analogue for Theorem II.3 for WOT convergence.

**Theorem II.4.** *Let $\widetilde{\mathcal{V}}$ be the set of functions $g \in L_1(\mathbb{T})$, so that*

$$\int_{\mathbb{T}} S_{\sigma,N}[f](t)g(t)dt \to \int_{\mathbb{T}} f(t)g(t)dt$$

*for all $f \in C(\mathbb{T})$, $g \in \widetilde{\mathcal{V}}$ and every bijection $\sigma\colon \mathbb{Z} \to \mathbb{Z}$. Then $\widetilde{\mathcal{V}} = L_2(\mathbb{T})$.*

## III. Fourier series on discrete groups

Let $\Gamma$ be a discrete countable group. Denote

$$\ell_2(\Gamma) = \{(c_\gamma)_{\gamma\in\Gamma} : \ c_\gamma \in \mathbb{C} \text{ and } \|c\|_{\ell_2(\Gamma)} = \sum_{\gamma\in\Gamma} |c_\gamma|^2 < \infty\}.$$

The set of all bounded linear operators from $\ell_2(\Gamma) \to \ell_2(\Gamma)$ is denoted by $\mathcal{B}(\ell_2(\Gamma))$. For every $\gamma \in \Gamma$, consider the shift operator $U_\gamma \in \mathcal{B}(\ell_2(\Gamma))$ defined by

$$(U_\gamma c)_h = c_{h\gamma^{-1}} \text{ for } (c_h)_{h\in\Gamma} \in \ell_2(\Gamma).$$

These shift operators induce what is called the left regular representation of $\Gamma$ on $\ell_2(\Gamma)$.

**Definition 2.** We say that $\{A_N\} \subset \mathcal{B}(\ell_2(\Gamma))$ converges to $A \in \mathcal{B}(\ell_2(\Gamma))$ in the strong operator topology if

$$A_N(g) \to A(g) \text{ in } \ell_2(\Gamma) \text{ for every } g \in \ell_2(\Gamma).$$

Likewise $\{A_N\} \subset \mathcal{B}(\ell_2(\Gamma))$ converges to $A \in \mathcal{B}(\ell_2(\Gamma))$ in the weak operator topology if

$$\langle A_N(g_1), g_2\rangle \to \langle A(g_1), g_2\rangle \text{ for every } g_1, g_2 \in \ell_2(\Gamma).$$

**Definition 3.** The smallest algebra closed in the operator norm topology and containing $\{U_\gamma\}_{\gamma\in\Gamma}$ is called the reduced group $C^*$–algebra of $\Gamma$ and is denoted by $C^*_r(\Gamma)$.

The smallest algebra closed in the SOT (or, equivalently in this case, WOT) and containing $\{U_\gamma\}_{\gamma\in\Gamma}$ is called the von Neumann algebra of $\Gamma$ and denoted by $L(\Gamma)$.

Note that $C_r^*(\Gamma) \subset L(\Gamma)$. For $A \in L(\Gamma)$, derived from the matrix representation of $A$ in the canonical basis $\{\delta_\gamma\}_{\gamma\in\Gamma}$ of $\ell_2(\Gamma)$, we can consider the formal series

$$\sum_{\gamma\in\Gamma} c_\gamma U_\gamma \tag{4}$$

where $c_\gamma = \langle A(\delta_\gamma), \delta_e\rangle$ and $e$ is the identity element of $\Gamma$. We use the notation

$$\mathrm{coef}(A) = (\langle A(\delta_\gamma), \delta_e\rangle)_{\gamma\in\Gamma}.$$

If $\Gamma$ is Abelian, then given a sequence $c = \{c_\gamma\}_{\gamma\in\Gamma} \in \ell_2(\Gamma)$, $c = \mathrm{coef}(A)$ for some $A \in C_r^*(\Gamma)$ if and only if there exists an $f \in C(\hat{\Gamma})$ such that $c_\gamma = \check{f}(\gamma)$ where $\hat{\Gamma}$ is the Pontryagin dual of $\Gamma$ and $\check{f}$ is the inverse Fourier transform of $f$ (see [13], Ch. III.1). In particular, when $\Gamma = \mathbb{Z}$, $U_n$ is the left shift operator on $\ell_2(\mathbb{Z})$ so via Fourier transform it corresponds to the multiplication operator by $e^{2\pi int}$ and $C_r^*(\mathbb{Z})$ corresponds to continuous functions on $\mathbb{T}$. In this case, $L(\mathbb{Z}) = L_\infty(\mathbb{T})$.

Ulyanov's problem in the context of reduced group $C^*-$algebras can be formulated as follows.

**Problem 3.** *Given $A \in C_r^*(\Gamma)$, is it possible to index the elements in $\Gamma$ by $\mathbb{N}$, $\Gamma = \{\gamma_n\}_{n=0}^\infty$, in such a way that*

$$\sum_{n=0}^{N} c_{\gamma_n} U_{\gamma_n} \to A$$

*in the operator norm where $c = \mathrm{coef}(A)$?*

To consider a relaxation of this inspired by Révész's Theorem, given $I \subset \Gamma$ with $|I| < \infty$, denote

$$S_I[A] = \sum_{\gamma\in I} c_\gamma U_\gamma.$$

**Definition 4.** We say that the group $\Gamma$ satisfies the Révész property if for every $A \in C_r^*(\Gamma)$ there exists a nested exhaustion of $\Gamma$, i.e. $\{I_N\}_{N\in\mathbb{N}}$, $I_N \subset I_{N+1}$, $|I_N| < \infty$ which satisfy $\bigcup_N I_N = \Gamma$ such that $S_{I_N}[A] \to A$ in the operator norm.

By Theorem I.1, the group of integers $\mathbb{Z}$ has the Révész property. The fact that $\mathbb{Z}^d$ has the Révész property was proved in [11]. The Révész property for discrete countable Abelian groups follows from the next theorem.

**Theorem III.1.** *The following permanence properties are true for the Révész property. Always $\Gamma$ will be a countable discrete group.*

1) *Suppose that $\Gamma$ has subgroups $(H_n)_{n\in\mathbb{N}}$ so that $H_n \subseteq H_{n+1}$ and $\Gamma = \bigcup_n H_n$. If each $H_n$ has the Révész property, then so does $\Gamma$.*
2) *If $\Gamma$ has the Révész property, then any subgroup of $\Gamma$ has the Révész property.*
3) *Suppose that $H$ is a finite index subgroup of $\Gamma$. If $H$ has the Révész property, then so does $\Gamma$.*

It is not clear whether every countable discrete non-abelian group has the Révész property; in fact we have the following two conjectures.

**Conjecture.** 1) *The free group of two elements $F_2$ does not have the Révész property;*
2) *Groups of polynomial growth have the Révész property.*

Since the proof of equivalences in Theorem II.1 relies on Révész's theorem, it is not obvious also whether the statement of Theorem II.1 holds for general groups or not.

For the analogue of Theorem II.3, let $\mathcal{V}$ be the set of all $g \in \ell_2(\Gamma)$ so that $\|S_{I_N}[A](g) - A(g)\|_{\ell_2(\Gamma)} \to 0$ for all $A \in C_r^*(\Gamma)$, $g \in \mathcal{V}$ and every exhaustion of $\Gamma$ given by $\{I_N\}_{N\in\mathbb{N}}$, $I_N \subset I_{N+1}$, $|I_N| < \infty$ and $\bigcup_N I_N = \Gamma$.

The analogue of the space $L_\infty(\mathbb{T})$ is the set

$$\mathcal{L}(\Gamma) = \{g = \mathrm{coef}(A) : \ A \in L(\Gamma)\}.$$

**Theorem III.2.** *The following statements are equivalent.*

1) *$\mathcal{V} = \mathcal{L}(\Gamma)$,*
2) *There is an absolute constant $c > 0$ so that for any projection $P \in L(\Gamma)$ with $P(e) > 0$, there is an $A \in C_r^*(\Gamma)$ with $\|a\|_{op} \leq 1$, and a subset $I \subseteq \Gamma$ so that $\|\mathrm{coef}(S_I[A]P)\|_{\ell_2(\Gamma)} \geq c$.*

However, unlike the integer case, we don't have a proof of condition 2) in Theorem III.2 for arbitrary groups.

Somewhat surprisingly, Theorem II.2 does not generalize to all non-commutative groups; however, the condition there turns out to give yet another characterization of amenability. Denote by $W(\Gamma)$ the set of $A \in C_r^*(\mathbb{T})$ for which $\mathrm{coef}(A) \in \ell_1(\Gamma)$; this is the natural analogue of Wiener algebra for general groups. Similar to Theorem II.2, the following holds.

**Theorem III.3.** *Let $\mathcal{W}$ be the set of all $A \in C_r^*(\Gamma)$ whose Fourier series converge unconditionally in SOT. Then $\mathcal{W} = W(\Gamma)$ if and only if $\Gamma$ is an amenable group.*

There are rather a lot of equivalent definitions of *amenable groups*, see e.g. [3, Section 2.6]. Perhaps the most tangible definition for our circumstances is that via Følner sequences.

**Definition 5.** Let $\Gamma$ be a countable, discrete group. A sequence $(F_n)_{n=1}^\infty$ of finite, nonempty subsets of $\Gamma$ is said to be a *Følner sequence* if

$$\lim_{n\to\infty} \frac{|\gamma F_n \Delta F_n|}{|F_n|} = 0, \text{ for all } \gamma \in \Gamma.$$

We say that $\Gamma$ is *amenable* if it has a Følner sequence.

Example of amenable groups include finite groups, abelian groups, solveable groups, niltpotent groups. See [2, Section G.2]. The class of amenable groups also possesses the following permanence properties (see [2, Appendix G]):

- if $\Gamma$ is amenable and $\Lambda$ is a subgroup of $\Gamma$, then $\Lambda$ is amenable;
- if $\Gamma$ is a group which has a finite-index amenable subgroup, then $\Gamma$ is amenable;
- if $\Gamma$ is amenable, then so is any quotient of $\Gamma$;

- if $\Lambda$ is a normal subgroup of a group $\Gamma$, and if $\Lambda, \Gamma/\Lambda$ are amenable, then $\Gamma$ is amenable;
- if $(\Gamma_n)_n$ are an increasing sequence of amenable subgroups of $\Gamma$, and if $\Gamma = \bigcup_n \Gamma_n$, then $\Gamma$ is amenable.

The class of amenable groups also include all groups of intermediate growth, such as the Grigorchuk group [5].

Some examples of groups which are *not* amenable include all nonabelian free groups, lattices in $SL_n(\mathbb{R}), SO(n,1)$, certain free Burnside groups, any free product $\Gamma * \Lambda$ provided $(|\Gamma| - 1)(|\Lambda| - 1) \geq 2$, and any non-elementary hyperbolic group. See [2, Appendix G],[1], [4].


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## References


[1] S. I. Adyan. Random walks on free periodic groups. *Izv. Akad. Nauk SSSR Ser. Mat.*, 46(6):1139–1149, 1343, 1982.

[2] B. Bekka, P. de la Harpe, and A. Valette. *Kazhdan's property (T)*, volume 11 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008.

[3] N. P. Brown and N. Ozawa. *$C^*$-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[4] É. Ghys and P. de la Harpe, editors. *Sur les groupes hyperboliques d'après Mikhael Gromov*, volume 83 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1990. Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988.

[5] R. I. Grigorchuk. Degrees of growth of finitely generated groups and the theory of invariant means. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(5):939–985, 1984.

[6] K. Hamm, B. Hayes, and A. Petrosyan. An operator theoretic approach to the convergence of rearranged fourier series. *arXiv preprint arXiv:1809.08217*, 2018.

[7] J. Kahane and Y. Katznelson. Sur les ensembles de divergence des séries trigonométriques. *Studia Mathematica*, 3(26):307–313, 1966.

[8] S. V. Konyagin. Rearrangements of trigonometric series and trigonometric polynomials. *Real Anal. Exchange*, 29(1):323–334, 2003.

[9] S. V. Konyagin. On uniformly convergent rearrangements of trigonometric Fourier series. *Journal of Mathematical Sciences*, 155(1):81–88, Nov 2008.

[10] J. D. McNeal and Y. E. Zeytuncu. A note on rearrangement of fourier series. *Journal of mathematical analysis and applications*, 323(2):1348–1353, 2006.

[11] S. G. Révész. On the convergence of Fourier series of U.A.P. functions. *Journal of Mathematical Analysis and Applications*, 151(2):308 – 317, 1990.

[12] S. G. Révész. Rearrangements of Fourier series. *Journal of Approximation Theory*, 60(1):101 – 121, 1990.

[13] M. Takesaki. *Theory of operator algebras II*, volume 125. Springer Science & Business Media, 2013.

[14] P. L. Ulyanov. Solved and unsolved problems in the theory of trigonometric and orthogonal series. *Russian Mathematical Surveys*, 19(1):1–62, 1964.