Ill-conditionedness of discrete Gabor phase retrieval and a possible remedy

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***Abstract*—In light of recent work on continuous Gabor phase retrieval, we analyse discrete Gabor phase retrieval problems and note that under realistic decay assumptions on the window functions, the stability constants increase significantly in the space dimension. When using discretisations of the Gaussian as windows, we are in fact able to show that the stability constants grow at least exponentially as the dimension of the space increases.**

At the same time, we observe that the adversarial examples, which we construct to estimate the stability constants, all contain long modes of silence. This suggests that one should try to reconstruct signals up to so-called semi-global phase factors and not up to a global phase factor as is the canon in the literature.

This observation is further corroborated by a stability result for discrete Gabor phase retrieval which we have proven recently.

***Index Terms*—Phase retrieval, short-time Fourier transform**

I. Introduction and basic notions

We consider the problem of *phase retrieval* which refers to the recovery of a signal (that is modelled by $\mathbf{x} \in \mathbb{C}^L$) from phaseless measurements. Applications of phase retrieval include ptychography for coherent diffraction imaging and audio processing. In 2016, it was shown that in the deterministic setting phase retrieval is always stable in finite dimensional Hilbert spaces yet always unstable in infinite dimensional Hilbert spaces [4].

More recently, the authors of [2] have analysed the instability in a phase retrieval problem on an infinite dimensional Hilbert space more closely. To be precise, they consider the *Gaussian window* $\psi \in L^2(\mathbb{R})$ given by $\psi(t) := \mathrm{e}^{-\pi t^2}$, $t \in \mathbb{R}$, along with the *Gabor transform* $\mathcal{V}_\psi : L^2(\mathbb{R}) \to \mathcal{C}(\mathbb{R} \times \mathbb{R}; \mathbb{C})$ given by

$$\mathcal{V}_\psi[f](t,\xi) := \int_{\mathbb{R}} f(s)\psi(s-t)\mathrm{e}^{-2\pi \mathrm{i} s\xi}\,\mathrm{d}s, \qquad t,\xi \in \mathbb{R},$$

for $f \in L^2(\mathbb{R})$. The Gabor phase retrieval problem is to reconstruct a signal $f \in L^2(\mathbb{R})$ from the magnitude measurement $|\mathcal{V}_\psi[f]|$. One should note that the signals $\mathrm{e}^{\mathrm{i}\alpha} f$, for $\alpha \in \mathbb{R}$, all generate exactly the same measurements $|\mathcal{V}_\psi[\mathrm{e}^{\mathrm{i}\alpha} f]| = |\mathcal{V}_\psi[f]|$. Therefore, it is the norm in the phase retrieval literature to attempt reconstruction of f *up to global phase* which essentially is the attempt to reconstruct an equivalence class $[f] := \{g \in L^2(\mathbb{R}) \mid g = \mathrm{e}^{\mathrm{i}\alpha} f,\ \alpha \in \mathbb{R}\}$.

In their analysis, the authors of [2] consider two signals $f_\lambda^\pm \in L^2(\mathbb{R})$, $\lambda > 0$, consisting of two Gaussian bumps each.

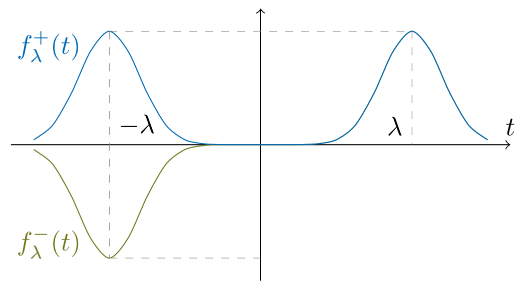

Fig. 1. The two signals $f_\lambda^\pm$.

More precisely, those signals are given by

$$f_\lambda^+(t) := \psi(t-\lambda) + \psi(t+\lambda),$$
$$f_\lambda^-(t) := \psi(t-\lambda) - \psi(t+\lambda),$$

with $\lambda > 0$ and $t \in \mathbb{R}$ (for a depiction see figure 1). It is not hard to see that

$$\inf_{\alpha \in \mathbb{R}} \left\| f_\lambda^+ - \mathrm{e}^{\mathrm{i}\alpha} f_\lambda^- \right\|_{L^2(\mathbb{R})} = 2^{\frac{3}{4}}.$$

With some effort, one can also show (see proposition 3.1 in [2]) that there exists a constant $c > 0$ such that for all $\lambda > 0$,

$$\left\| |\mathcal{V}_\psi[f_\lambda^+]| - |\mathcal{V}_\psi[f_\lambda^-]| \right\|_{W^{1,2}(\mathbb{R})} \leq c\mathrm{e}^{-\lambda^2}.$$

In particular, the recovery of a signal $f \in L^2(\mathbb{R})$ (up to global phase) from $|\mathcal{V}_\psi[f]|$ is in some sense *severely ill-posed*.

In [1], the authors propose to overcome this ill-posedness of the Gabor phase retrieval problem by recovering signals $f \in L^2(\mathbb{R})$ up to multiple so-called *semi-global* phase factors (and thus not necessarily up to a global phase factor). In the case of the function f_λ^+, where $\lambda \in \mathbb{R}$ is fixed, one would, for instance, not try to recover $[f_\lambda^+] = \{g \in L^2(\mathbb{R}) \mid g = \mathrm{e}^{\mathrm{i}\alpha} f_\lambda^+,\ \alpha \in \mathbb{R}\}$ but $\{g \in L^2(\mathbb{R}) \mid g = \mathrm{e}^{\mathrm{i}\alpha} f_\lambda^\sigma,\ \alpha \in \mathbb{R},\ \sigma \in \{+,-\}\}$. So both of the Gaussian bumps in f_λ^+ will be recovered up to global phase whereas f_λ^+ will be recovered up to semi-global phase. In [1], the semi-globality is formalised using so-called *atoll functions* and the authors prove that phase retrieval up to semi-global phase from measurements generated by the Gabor transform with Gaussian window is stable in $L^2(\mathbb{R})$.

Our goal in this paper is to discretise the findings of [2] and thus motivate the use of semi-global phase factors in discrete phase retrieval problems. To be precise, we will show in section II that the stability constants for discrete Gabor phase retrieval problems can increase exponentially in the space dimension and that therefore discrete Gabor phase retrieval is ill-conditioned. In section III, we will formalise semi-globality with the help of graph theory and present a related result of ours [3] which emphasises the usefulness of semi-globality in the discrete setting. The rest of this first section is dedicated to the introduction of certain basic notions necessary to the exposition. We finally remark that it is not our goal to provide statements about tractable algorithms or to give insights on how to develop such algorithms in this paper.

A. The discrete Gabor transform

Let $L \in \mathbb{N}$. On the space of signals $\mathbb{C}^L$, we can introduce the *discrete Fourier transform (DFT)* $\mathcal{F}^{\mathrm{d}} : \mathbb{C}^L \to \mathbb{C}^L$ by

$$\mathcal{F}_{\mathrm{d}}[\mathbf{x}](n) := \frac{1}{\sqrt{L}} \cdot \sum_{\ell=0}^{L-1} \mathbf{x}(\ell) \mathrm{e}^{-2\pi \mathrm{i} \frac{\ell n}{L}}, \quad n = 0, \ldots, L-1,$$

for $\mathbf{x} \in \mathbb{C}^L$. It can be shown that the inverse of the DFT satisfies that for all $\boldsymbol{\xi} \in \mathbb{C}^L$,

$$\mathcal{F}_{\mathrm{d}}^{-1}[\boldsymbol{\xi}](\ell) = \frac{1}{\sqrt{L}} \cdot \sum_{n=0}^{L-1} \boldsymbol{\xi}(n) \mathrm{e}^{2\pi \mathrm{i} \frac{n\ell}{L}}, \quad \ell = 0, \ldots, L-1.$$

For a window function $\boldsymbol{\phi} \in \mathbb{C}^L$, we can furthermore introduce the *discrete Gabor transform (DGT)* $\mathcal{V}_{\boldsymbol{\phi}}^{\mathrm{d}} : \mathbb{C}^L \to \mathbb{C}^{L \times L}$ for $\mathbf{x} \in \mathbb{C}^L$ by

$$\mathcal{V}_{\boldsymbol{\phi}}^{\mathrm{d}}[\mathbf{x}](m,n) := \frac{1}{\sqrt{L}} \cdot \sum_{\ell=0}^{L-1} \mathbf{x}(\ell) \overline{\boldsymbol{\phi}(\ell - m)} \mathrm{e}^{-2\pi \mathrm{i} \frac{\ell n}{L}},$$

with $m, n \in \{0, \ldots, L-1\}$ and where the indexing of $\boldsymbol{\phi}$ (and every other vector in this paper) is understood to be evaluated modulo L. One handy way of writing the DGT is by introducing the linear operators $\Pi_{(m,n)} : \mathbb{C}^L \to \mathbb{C}^L$, for $m, n \in \{0, \ldots, L-1\}$, as

$$\Pi_{(m,n)}[\mathbf{x}](\ell) := \frac{1}{\sqrt{L}} \mathbf{x}(\ell - m) \mathrm{e}^{2\pi \mathrm{i} \frac{\ell n}{L}}, \quad \ell = 0, \ldots, L-1,$$

where $\mathbf{x} \in \mathbb{C}^L$, and writing

$$\mathcal{V}_{\boldsymbol{\phi}}^{\mathrm{d}}[\mathbf{x}](m,n) = \left(\mathbf{x}, \Pi_{(m,n)}[\boldsymbol{\phi}]\right),$$

where $(\cdot, \cdot)$ denotes the Euclidean scalar product on $\mathbb{C}^L$.

The *discrete Gabor phase retrieval problem* can now be posed as the recovery of a signal $\mathbf{x} \in \mathbb{C}^L$ from the phaseless measurements

$$M_{\boldsymbol{\phi}}[\mathbf{x}](m,n) := \left|\mathcal{V}_{\boldsymbol{\phi}}^{\mathrm{d}}[\mathbf{x}](m,n)\right|^2, \tag{1}$$

where $m, n \in \{0, \ldots, L-1\}$. Similar to the Gabor phase retrieval problem, we will not be able to distinguish the different signals $\mathrm{e}^{\mathrm{i}\alpha}\mathbf{x}$, for $\alpha \in \mathbb{R}$, from the measurements $M_{\boldsymbol{\phi}}[\mathrm{e}^{\mathrm{i}\alpha}\mathbf{x}] = M_{\boldsymbol{\phi}}[\mathbf{x}]$ and will therefore view the *phase retrieval operator* $M_{\boldsymbol{\phi}}$ as a map on the quotient space $\mathbb{C}^L / \mathcal{S}^1$, where $\mathcal{S}^1 := \{\mathrm{e}^{\mathrm{i}\alpha} \mid \alpha \in \mathbb{R}\}$ denotes the *unit circle*.

Our results in this paper rely heavily on the following formula which is well-known in the literature.

Lemma 1 *Let $L \in \mathbb{N}$ and denote by $* : \mathbb{C}^L \times \mathbb{C}^L \to \mathbb{C}^L$ the discrete convolution given by*

$$(\mathbf{x} * \mathbf{y})(k) := \sum_{\ell=0}^{L-1} \mathbf{x}(\ell) \mathbf{y}(k - \ell), \quad k = 0, \ldots, L-1,$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^L$. Let also $\mathbf{x}, \boldsymbol{\phi} \in \mathbb{C}^L$, $m, k \in \{0, \ldots, L-1\}$ and define $\mathbf{x}_m, \mathbf{x}_m^{\#} \in \mathbb{C}^L$ as

$$\mathbf{x}_m(\ell) := \mathbf{x}(\ell)\overline{\boldsymbol{\phi}(\ell - m)} \quad \textit{and} \quad \mathbf{x}_m^{\#}(\ell) := \overline{\mathbf{x}_m(-\ell)}$$

for all $\ell \in \{0, \ldots, L-1\}$. Then, the measurements $M_{\boldsymbol{\phi}}[\mathbf{x}]$ (cf. (1)*) satisfy the* autocorrelation relation

$$\mathcal{F}_{\mathrm{d}}^{-1}\left[M_{\boldsymbol{\phi}}[\mathbf{x}](m, \cdot)\right](k) = \frac{1}{\sqrt{L}} \cdot \left(\mathbf{x}_m * \mathbf{x}_m^{\#}\right)(k).$$

Remark 2 *This well-known relation has been used in multiple different papers [5], [7], [9] to deduce under what circumstances the operator $M_{\boldsymbol{\phi}} : \mathbb{C}^L / \mathcal{S}^1 \to \mathbb{R}^{L \times L}$ is injective.*

B. A characterisation of decay

In practice, the majority of utilised window functions $\boldsymbol{\phi} \in \mathbb{C}^L$ have some decay properties. In this subsection, we want to formalise what is meant by decay.

Definition 3 *Consider a family of window functions $\Phi := \{\boldsymbol{\phi}_L \mid \boldsymbol{\phi}_L \in \mathbb{C}^L,\ L \in \mathbb{N}\} \subset \bigcup_{L \in \mathbb{N}} \mathbb{C}^L$.*

i. *We call Φ a family of* polynomially decaying *window functions if there exist $L_0 \in \mathbb{N}$, $c > 0$ and $\kappa \in \mathbb{R}$ such that for all $\ell, L \in \mathbb{N}$, with $L \geq L_0$ and $L_0 \leq \ell + 1 \leq L$,*

$$|\boldsymbol{\phi}_L(\ell)| \leq c\ell^{-\kappa}.$$

ii. *We call Φ a family of* superpolynomially decaying *window functions if for all $\kappa \in \mathbb{R}$, there exist $L_0 \in \mathbb{N}$ and $c > 0$ such that for all $\ell, L \in \mathbb{N}$, with $L \geq L_0$ and $L_0 \leq \ell + 1 \leq L$,*

$$|\boldsymbol{\phi}_L(\ell)| \leq c\ell^{-\kappa}.$$

iii. *We call Φ a family of* exponentially decaying *window functions if there exist $L_0 \in \mathbb{N}$, $c > 0$ and $\lambda > 0$ such that for all $\ell, L \in \mathbb{N}$, with $L \geq L_0$ and $L_0 \leq \ell + 1 \leq L$,*

$$|\boldsymbol{\phi}_L(\ell)| \leq c\mathrm{e}^{-\lambda \ell}.$$

Example 4 *Let $N \in \mathbb{N}$ and $\boldsymbol{\phi} \in \mathbb{C}^N$. For $L \in \mathbb{N}$, define $\boldsymbol{\phi}_L \in \mathbb{C}^L$ by*

$$\boldsymbol{\phi}_L(\ell) := \begin{cases} \boldsymbol{\phi}(\ell) & \textit{if } \ell = 0, \ldots, N-1, \\ 0 & \textit{else}, \end{cases}$$

to obtain a family of window functions $\{\boldsymbol{\phi}_L\}_{L \in \mathbb{N}}$ that is exponentially decaying with any $c > 0$, $L_0 = N + 2$ and any $\lambda > 0$.

Remark 5 (From the perspective of audio processing) *In audio processing, the signal $\mathbf{x} \in \mathbb{C}^L$ is given by a digitalisation of an audio signal. The length L of $\mathbf{x}$ is determined by two factors: The sampling rate $f \in \mathbb{N}$ which is typically given in Hertz and the temporal length of the audio signal $T > 0$ which is typically given in seconds. In such a* *setup the digitalisation of the audio signal* $\mathbf{x}$ *would have the length* $L = \lfloor f \cdot T \rfloor \in \mathbb{N}$.

To mimic the result from [2], we may fix a sampling rate $f \in \mathbb{N}$ *and consider a family of Gaussian windows satisfying that for all* $L \in \mathbb{N}$,

$$\boldsymbol{\phi}_L(\ell) := \mathrm{e}^{-\pi \cdot \left(\frac{\ell - f}{f}\right)^2}, \qquad \ell = 0, \ldots, L-1.$$

In this way, we obtain a family of window functions that is exponentially decaying (with $c = \mathrm{e}^{\pi}$, $L_0 = 2f+1$ *and any* $\lambda \in (0, \pi/f]$*) in which each individual window function has the same full width at half maximum* $2\sqrt{\log 2/\pi} f$ *which corresponds to* $2\sqrt{\log 2/\pi}$ *seconds.*

II. Severe ill-conditionedness of discrete Gabor phase retrieval

In this section, we want to deduce asymptotic bounds for the stability constants of discrete Gabor phase retrieval problems.

Definition 6 (Stability constant) *Let* $L \in \mathbb{N}$. *The* stability constant *of the* L*-dimensional discrete Gabor phase retrieval problem is the smallest* $c_L > 0$ *for which we have that all* $\mathbf{x}, \mathbf{y} \in \mathbb{C}^L$ *satisfy (cf. (1))*

$$\inf_{\alpha \in \mathbb{R}} \left\| \mathbf{x} - \mathrm{e}^{\mathrm{i}\alpha} \mathbf{y} \right\|_2 \leq c_L \cdot \left\| M_{\boldsymbol{\phi}}[\mathbf{x}] - M_{\boldsymbol{\phi}}[\mathbf{y}] \right\|_{\mathrm{F}},$$

where $\|\cdot\|_2$ *denotes the Euclidean norm and* $\|\cdot\|_{\mathrm{F}}$ *denotes the Frobenius norm.*

We can show that for a family Φ of exponentially decaying window functions, the stability constants increase exponentially in the space dimension L and that therefore the corresponding phase retrieval problem is severely ill-conditioned.

Theorem 7 (Main Result) *Let* $C > 0$ *and let* Φ *be a family of exponentially decaying window functions such that for all* $L \in \mathbb{N}$, $\|\boldsymbol{\phi}_L\|_\infty < C$. *Then, the stability constants* $\{c_L\}_{L \in \mathbb{N}}$ *of the corresponding discrete Gabor phase retrieval problems form an (at least) exponentially increasing sequence. More precisely, there exist* $L_0 \in \mathbb{N}$, $c > 0$ *and* $\lambda > 0$ *such that for all* $L \in \mathbb{N}$, *with* $L \geq L_0$,

$$c_L \geq c\mathrm{e}^{\lambda L}$$

holds.

Proof Let $L \in \mathbb{N}$ and consider $\mathbf{x}_L^{\pm} \in \mathbb{R}^L$ given by

$$\mathbf{x}_L^{\pm}(\ell) := \begin{cases} 1 & \text{if } \ell = 0, \\ \pm 1 & \text{if } \ell = \left\lfloor \frac{L}{2} \right\rfloor, \\ 0 & \text{else.} \end{cases}$$

Then,

$$\inf_{\alpha \in \mathbb{R}} \left\| \mathbf{x}_L^+ - \mathrm{e}^{\mathrm{i}\alpha} \mathbf{x}_L^- \right\|_2 = \left\| \mathbf{x}_L^+ - \mathbf{x}_L^- \right\|_2 = 2.$$

By definition of the Frobenius norm and Plancherel's theorem, we have

$$\begin{aligned} &\left\| M_{\boldsymbol{\phi}_L}[\mathbf{x}_L^+] - M_{\boldsymbol{\phi}_L}[\mathbf{x}_L^-] \right\|_{\mathrm{F}}^2 \\ &\quad = \sum_{m=0}^{L-1} \left\| \mathcal{F}_{\mathrm{d}}^{-1} \left[M_{\boldsymbol{\phi}_L}[\mathbf{x}_L^+](m, \cdot) - M_{\boldsymbol{\phi}_L}[\mathbf{x}_L^-](m, \cdot) \right] \right\|_2^2. \end{aligned}$$

By lemma 1, we further obtain

$$\begin{aligned} &\left\| M_{\boldsymbol{\phi}_L}[\mathbf{x}_L^+] - M_{\boldsymbol{\phi}_L}[\mathbf{x}_L^-] \right\|_{\mathrm{F}}^2 \\ &\qquad = \frac{1}{L} \cdot \sum_{m=0}^{L-1} \left\| \mathbf{x}_{L,m}^+ * \mathbf{x}_{L,m}^{+,\#} - \mathbf{x}_{L,m}^- * \mathbf{x}_{L,m}^{-,\#} \right\|_2^2, \end{aligned}$$

where

$$\mathbf{x}_{L,m}^{\pm}(\ell) := \mathbf{x}_L^{\pm}(\ell) \overline{\boldsymbol{\phi}_L(\ell - m)} \quad \text{and} \quad \mathbf{x}_{L,m}^{\pm,\#}(\ell) := \overline{\mathbf{x}_{L,m}^{\pm}(-\ell)}$$

for $\ell, m \in \{0, \ldots, L-1\}$. Finally, for $k, m \in \{0, \ldots, L-1\}$, the following relation holds:

$$\begin{aligned} &\left(\mathbf{x}_{L,m}^{\pm} * \mathbf{x}_{L,m}^{\pm,\#} \right)(k) \\ &\qquad = \sum_{\ell=0}^{L-1} \mathbf{x}_L^{\pm}(\ell) \overline{\mathbf{x}_L^{\pm}(\ell - k)} \boldsymbol{\phi}_L(\ell - k - m) \overline{\boldsymbol{\phi}_L(\ell - m)}. \end{aligned}$$

Thus, if L is even,

$$\begin{aligned} &\left(\mathbf{x}_{L,m}^{\pm} * \mathbf{x}_{L,m}^{\pm,\#} \right)(k) \\ &\qquad = \begin{cases} \left| \boldsymbol{\phi}_L(-m) \right|^2 + \left| \boldsymbol{\phi}_L\left(\frac{L}{2} - m\right) \right|^2 & \text{if } k = 0, \\ \pm 2\Re\left[\boldsymbol{\phi}_L(-m) \overline{\boldsymbol{\phi}_L\left(\frac{L}{2} - m\right)} \right] & \text{if } k = \frac{L}{2}, \\ 0 & \text{else,} \end{cases} \end{aligned}$$

and if L is odd,

$$\begin{aligned} &\left(\mathbf{x}_{L,m}^{\pm} * \mathbf{x}_{L,m}^{\pm,\#} \right)(k) \\ &\qquad = \begin{cases} \left| \boldsymbol{\phi}_L(-m) \right|^2 + \left| \boldsymbol{\phi}_L\left(\left\lfloor\frac{L}{2}\right\rfloor - m\right) \right|^2 & \text{if } k = 0, \\ \pm \boldsymbol{\phi}_L(-m) \overline{\boldsymbol{\phi}_L\left(\left\lfloor\frac{L}{2}\right\rfloor - m\right)} & \text{if } k = \left\lfloor\frac{L}{2}\right\rfloor, \\ \pm \boldsymbol{\phi}_L\left(\left\lfloor\frac{L}{2}\right\rfloor - m\right) \overline{\boldsymbol{\phi}_L(-m)} & \text{if } k = \left\lceil\frac{L}{2}\right\rceil \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned} &\left\| \mathbf{x}_{L,m}^+ * \mathbf{x}_{L,m}^{+,\#} - \mathbf{x}_{L,m}^- * \mathbf{x}_{L,m}^{-,\#} \right\|_2^2 \\ &\qquad \leq 16 \left| \boldsymbol{\phi}_L(-m) \boldsymbol{\phi}_L\left(\left\lfloor \frac{L}{2} \right\rfloor - m \right) \right|^2. \end{aligned}$$

As Φ is exponentially decaying, there exist $L_0 \in \mathbb{N}$, $c > 0$ and $\lambda > 0$ such that for all $\ell, L \in \mathbb{N}$, with $L \geq L_0$ and $L_0 \leq \ell + 1 \leq L$,

$$|\boldsymbol{\phi}_L(\ell)| \leq c\mathrm{e}^{-\lambda \ell}.$$

If we choose $L \geq 2L_0$, then $\lfloor L/2 \rfloor + 1 \geq L_0$ and either $m = 0$,

$$\left\lfloor \frac{L}{2} \right\rfloor \leq L - m < L \quad \text{or} \quad \left\lfloor \frac{L}{2} \right\rfloor < L + \left\lfloor \frac{L}{2} \right\rfloor - m < L.$$

Therefore, we have that

$$\begin{aligned} &\left\| \mathbf{x}_{L,m}^+ * \mathbf{x}_{L,m}^{+,\#} - \mathbf{x}_{L,m}^- * \mathbf{x}_{L,m}^{-,\#} \right\|_2^2 \\ &\qquad \leq 16c^2 \left\| \boldsymbol{\phi}_L \right\|_\infty^2 \cdot \mathrm{e}^{-\lambda(L-1)} \end{aligned}$$

and

$$\left\| M_{\boldsymbol{\phi}_L}[\mathbf{x}_L^+] - M_{\boldsymbol{\phi}_L}[\mathbf{x}_L^-] \right\|_{\mathrm{F}}^2 \leq 4cC\mathrm{e}^{\frac{\lambda}{2}} \cdot \mathrm{e}^{-\frac{\lambda}{2} L}$$

which proves the theorem. □

In a similar fashion, one may show that for a family Φ of polynomially decaying window functions, the stability constants increase polynomially in the space dimension L and for a family of superpolynomially decaying window functions, the stability constants increase superpolynomially in L.

Theorem 8 *Let $C > 0$, Φ a family of window functions such that for all $L \in \mathbb{N}$, $\|\phi_L\|_\infty < C$, and let $\{c_L\}_{L\in\mathbb{N}}$ be the stability constants of the corresponding discrete Gabor phase retrieval problems.*

i. *If Φ is polynomially decaying, then $\{c_L\}$ increases (at least) polynomially.*
ii. *If Φ is superpolynomially decaying, then $\{c_L\}$ increases (at least) superpolynomially.*

Remark 9 (On the Cauchy wavelet transform) *Similar results may also be proven for the problem of phase retrieval from discrete Cauchy wavelet transform measurements as it is presented in [8].*

III. A POSSIBLE REMEDY

The signals $\mathbf{x}_L^\pm \in \mathbb{C}^L$ used in the proof of theorem 7 notably contain a long mode of silence (when L is large). This in unison with the findings in [1], [2], [6] suggests that phase retrieval from discrete Gabor measurements might benefit from using a semi-global stability regime. Let us formalise semi-globality in this section and present a result from related work [3] which shows that discrete Gabor transform phase retrieval is stable in the semi-global setting.

To formalise semi-globality, we will introduce a graph capturing properties of the signals which we want to reconstruct. Let $L \in \mathbb{N}$, $\mathbf{x},\mathbf{y} \in \mathbb{C}^L$, $\delta_0 > 0$ and $\Delta \in \{0,\dots,\lfloor L/2\rfloor - 1\}$. Define the graph $G = (V,E)$ by letting the vertex set be defined by

$$V := \{\ell \in \{0,\dots,L-1\} \,|\, |\mathbf{x}(\ell)|, |\mathbf{y}(\ell)| > \delta_0\} \tag{2}$$

and placing an edge between two vertices $\ell,\ell' \in V$ if

$$1 \le |\ell - \ell'| \le \Delta + 1 \mod L. \tag{3}$$

In general, the graph G will consist of $K \in \{1,\dots,L-1\}$ connected components whose vertex sets we might denote by $\{V_k\}_{k=1}^K$. Then, we say that $\mathbf{x}$ agrees with $\mathbf{y}$ up to *semi-global phase* if for every connected component $V_k \in \{V_k\}_{k=1}^K$, there exists an $\alpha_k \in \mathbb{R}$ such that for all $\ell \in V_k$,

$$\mathbf{x}(\ell) = \mathrm{e}^{\mathrm{i}\alpha_k}\mathbf{y}(\ell).$$

Very recently, we have managed to prove a stability result for phase retrieval from discrete Gabor transform measurements based around semi-global phase reconstruction.

Theorem 10 (Theorem 3.5 in [3], p. 10) *Let $L \in \mathbb{N}$. Consider two signals $\mathbf{x},\mathbf{y} \in \mathbb{C}^L$, a window $\boldsymbol{\phi} \in \mathbb{C}^L$, two tolerance parameters $\delta_0,\delta_1 > 0$ and a maximum time separation parameter $\Delta \in \{0,\dots,\lfloor L/2\rfloor - 1\}$. Let $G = (V,E)$ be defined with V as in (2) and E defined through (3) and assume that G has K connected components with vertex sets $\{V_k\}_{k=1}^K$. Then,*

$$\begin{aligned}
&\inf_{\alpha_1,\dots,\alpha_K \in \mathbb{R}} \sum_{k=1}^K \left\|\mathbf{x} - \mathrm{e}^{\mathrm{i}\alpha_k}\mathbf{y}\right\|_{\ell^2(V_k)} \\
&\quad\le \frac{1}{\delta_0\delta_1}\cdot\left(\frac{1}{2} + \frac{\min\{\|\mathbf{x}\|_\infty, \|\mathbf{y}\|_\infty\}}{\delta_0}\cdot\sum_{k=1}^K |V_k|\right) \\
&\qquad\qquad \cdot \|M_{\boldsymbol{\phi}}[\mathbf{x}] - M_{\boldsymbol{\phi}}[\mathbf{y}]\|_{\mathrm{F}} \\
&\qquad + \frac{1}{\delta_0}\cdot\left(\frac{1}{2} + \frac{\min\{\|\mathbf{x}\|_\infty, \|\mathbf{y}\|_\infty\}}{\delta_0}\cdot\sum_{k=1}^K |V_k|\right)\cdot\epsilon
\end{aligned}$$

holds, with $\epsilon > 0$ satisfying

$$\frac{\epsilon^2}{2} = \sum_{k=0}^{\Delta+1} \sum_{\substack{\ell=0 \\ |(\boldsymbol{\phi},\Pi_{(k,\ell)}\boldsymbol{\phi})|\le\delta_1}}^{L-1} |(\mathbf{x},\Pi_{(k,\ell)}\mathbf{x}) - (\mathbf{y},\Pi_{(k,\ell)}\mathbf{y})|^2.$$

So phase retrieval from Gabor transform measurements up to semi-global phase is stable with stability constant

$$\frac{1}{\delta_0\delta_1}\cdot\left(\frac{1}{2} + \frac{\min\{\|\mathbf{x}\|_\infty, \|\mathbf{y}\|_\infty\}}{\delta_0}\cdot\sum_{k=1}^K |V_k|\right).$$

Here, the inverse dependence on δ_0 reflects the inherent instability of extracting phase information from complex numbers with small magnitude, the dependence on the cardinality of the vertex sets $\{V_k\}_{k=1}^K$ should be seen as a mild increase of the stability constant in the space dimension L as for a signal $\mathbf{x} \in \mathbb{C}^L$, with $|\mathbf{x}(\ell)| > \delta_0$, for all $\ell \in \{0,\dots,L-1\}$, we have $\sum_{k=1}^K |V_k| = L$, and the inverse dependence of the stability constant on δ_1 should be noted to be strongly connected to the error term ϵ. In fact, δ_1 and ϵ reflect a tradeoff between using information on the window function with small absolute value and thereby increasing the stability constant and neglecting information on the signals where the corresponding information on the window function is small and thereby increasing the error ϵ.

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