Deterministic guarantees for L^1-reconstruction: A large sieve approach with geometric flexibility

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***Abstract* – We present estimates of the p-concentration ratio for various function spaces on different geometries including the line, the sphere, the plane, and the hyperbolic disc, using large sieve methods. Thereby, we focus on L^1-estimates which can be used to guarantee the reconstruction from corrupted or partial information.**

I. Introduction

Consider a measure space (X, μ), a measurable subset $\Omega \subset X$ and a subspace of $\mathcal{S} \subset L^p(X, \mu)$. A fundamental quantity in mathematical signal analysis is the p-concentration ratio, defined as

$$\lambda_p(\Omega, \mathcal{S}) := \sup_{f \in \mathcal{S}} \frac{\|f \cdot \chi_\Omega\|_p^p}{\|f\|_p^p}. \qquad (1)$$

For $p = 2$, the study of this quantity was the cornerstone of the body of work nowadays referred to as the 'Bell Lab papers' of Landau, Slepian and Pollak, culminating in Landau's necessary conditions on sampling and interpolation [15]. This was later extended to a variety of contexts, including spaces of analytic functions [19], and wavelet and Gabor spaces [10], [11].

For $p = 1$, $\lambda_1(\Omega, \mathcal{S}) < \frac{1}{2}$ implies that

$$\|f \cdot \chi_\Omega\|_1 < \frac{1}{2}\|f\|_1, \qquad (2)$$

meaning that *every signal $f \in \mathcal{S}$ is sparse (poorly concentrated) in* Ω. Under such conditions, a remarkable phenomenon discovered by Logan [16], [7], holds: if we sense a signal f corrupted by *unknown* noise N supported in an *unknown* Ω, then f can be perfectly reconstructed as the solution of the L^1-minimization problem

$$f = \arg\min_{s \in \mathcal{S}} \|(f + N) - s\|_1.$$

Another reconstruction scenario can be derived by generalizing an observation of Donoho and Stark for bandlimited discrete functions [7, Theorem 9]. In the absence of noise, if we only sense the projection of a general $f \in S$ on Ω^c, then f can be perfectly reconstructed as the solution of the L^1-minimization problem

$$f = \arg\min_{s \in \mathcal{S}} \|s\|_1, \qquad \text{subject to } P_{\Omega^c} s = P_{\Omega^c} f.$$

The above scenarios offer the possibility of reconstructing a function from highly incomplete information, at the cost of obtaining a constant such that $\|f \cdot \chi_\Omega\|_1 < C\|f\|_1$, with $C \leq \frac{1}{2}$. This is in general extremely difficult and the only sharp result in this direction is provided by Tao's uncertainty principle for signals of prime lenght [21]. In Section 7.3 of [7], it has been suggested that considering a random Ω, one could improve the estimate $\lambda_1(\Omega, \mathcal{S})$. The full potential of this suggestion has been later explored in the groundbreaking papers of Donoho [9] and Candés, Romberg and Tao [6], where it has been shown that selecting both Ω and $\mathcal{S}$ randomly, one could obtain $C \leq \frac{1}{2}$ with high probability, under reasonable assumptions on the measure of Ω. This lead to an intense research activity on the topic nowadays known as Compressive Sensing [12].

We will pursue a different strategy for the estimation of the quantity $\lambda_1(\Omega, \mathcal{S})$, suggested by a ingenious application of the large sieve principle to sparse recovery problems by Donoho and Logan [8]. This work was inspired by the following variation of the classical large sieve inequality, due to Bombieri and quoted by Montgomery in [18, p. 562]. Let μ be a positive, periodic measure on the circle and

$f(x) = \sum_{k=m+1}^{m+n} a_k e^{2\pi i k x}$ a trigonometric polynomial. Then the following holds for every $0 < \delta \leq 1$:

$$\|f\|^2_{L^2[(0,1),\mu]} \leq \left(n + \frac{2}{\delta}\right)\left(\sup_{\alpha} \int_{\alpha}^{\alpha+\delta} d\mu\right) \|f\|^2_{L^2(0,1)} . \quad (3)$$

In [8], L^1-versions of (3) are obtained, assuring that (2) holds if the measure μ is sparse (low concentration on sets $[\alpha, \alpha+\delta]$). This lead us to pursue the program of obtaining large sieve inequalities of the Donoho-Logan type for $1 \leq p < \infty$, in several signal analytic settings where the original large sieve methods lack key ingredients (for instance, Beurling's theory of extremal functions). In this note we will outline the first results of this program. For convenience of presentation, consider the general setting of the first paragraph. Denote by $\mathcal{S}_K$ the reproducing kernel subspace of $L^p(X, \mu)$ consisting of all functions f such that

$$f(x) = \int_X K(x,y) f(y) d\mu(y), \quad \forall x \in X, \qquad (4)$$

for some Hermitian kernel K. We assume that X is a metric space, and that the kernel satisfies

$$\sup_{y \in X} \int_X |K(x,y)| d\mu(x) < \infty.$$

Then, for $1 \leq p < \infty$, every $f \in \mathcal{S}_K$ satisfies the following inequality:

$$\|f \cdot \chi_\Omega\|_p^p \leq \sup_{y \in X} \int_\Omega |K(x,y)| d\mu(x) \cdot \|f\|_p^p. \qquad (5)$$

This is a simple consequence of the reproducing kernel equation (4), since

$$\begin{aligned} \int_\Omega |f(x)| d\mu(x) &\leq \int_\Omega \int_X |K(x,y) f(y)| d\mu(y) d\mu(x) \\ &\leq \sup_{y \in X} \int_\Omega |K(x,y)| d\mu(x) \cdot \|f\|_1. \end{aligned}$$

The statement for general $1 \leq p < \infty$ then follows from the Riesz-Thorin theorem using the trivial observation that $\|f \cdot \chi_\Omega\|_\infty \leq \|f\|_\infty$.

From inequality (5) we conclude that (2) holds as long we can assure that

$$\sup_{y \in X} \int_\Omega |K(x,y)| d\mu(x) < C \leq \frac{1}{2}.$$

This is, of course, completely impossible to do in the absence of more information. However, in several situations where the kernel $K(x,y)$ is explicitly known, one can obtain large sieve type inequalities which can be used to obtain useful estimates from (5).

We will see some examples in the sections below, where the estimates are given in terms of a measure of the sparsity of the set Ω involving the quantity

$$\sup_{x \in X} \frac{|\Omega \cap B_\varrho(x,R)|}{|B_\varrho(x,R)|}, \qquad (6)$$

with $B_\varrho(x,R)$ the ball centered at x, measured in the metric ϱ. Here, ϱ is fine tuned to the underlying geometry of the space X, and R is chosen according to the space $\mathcal{S}_K$. If the p-concentration ratio is estimated by a constant times the expression in (6), it follows that functions in $\mathcal{S}_K$ cannot be well concentrated on 'sparse' sets, i.e. on sets that are of small size locally, and that any function in $\mathcal{S}_K$ can be separated from noise concentrated on such a set. We will see examples involving the line, planar euclidean, planar hyperbolic and spherical geometries. Due to the applications in L^1-minimization, we will mostly focus on estimates of $\lambda_1(\Omega, \mathcal{S})$, but the methods we use work for general $1 \leq p < \infty$, except for the spherical case, where the problem for $p = 1$ is still open. Nevertheless, we will also present a $p = 2$ large sieve bound for finite spherical harmonic expansions, which may be useful in different applications, taking into account the recent applications of the $p = 2$ large sieve bounds in superresolution on the so called well-separated case [17], [5].

The results are presented in the following sequence. For reference we start with one of the Donoho-Logan's large sieve inequalities for band-limited functions and then present the main result of the setting of finite spherical harmonics expansions. We then move to phase-space contexts and outline the results for Gabor spaces from [3], [4] and a new result for Bergman spaces which can be translated to the setting of Cauchy wavelets.

II. Concrete Large Sieve Inequalities

A. Donoho-Logan's Large Sieve for the Paley-Wiener Space

Consider the Paley-Wiener space of band-limited functions

$$PW_W^p := \left\{ f \in L^p(\mathbb{R}) : \ \mathrm{supp}(\hat{f}) \subseteq [-\pi W, \pi W] \right\},$$

where we use the following convention for the Fourier transform

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i \omega t} dt.$$

In [8], Donoho and Logan introduced the following notion of maximum Nyquist density:

$$\rho_{\mathbb{R}}(\Omega, W) = |W| \cdot \sup_{t\in\mathbb{R}} \left|\Omega \cap [t, t+1/W]\right|,$$

and obtained the large sieve inequality

$$\|f \cdot \chi_\Omega\|_1 \le \frac{\pi}{2} \cdot \rho_{\mathbb{R}}(\Omega, W) \cdot \|f\|_1, \quad \forall f \in PW_W^1.$$

This shows that $\rho_{\mathbb{R}}(\Omega, W) < \frac{1}{\pi}$ is enough to assure perfect recovery in the context outlined in the introduction. The results in [8] also cover discrete settings and applications of the $p = 2$ inequality.

B. Finite Spherical Harmonics Expansions

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$ and $\mathcal{S}_L$ be the space of finite spherical harmonics expansions of maximum degree L, i.e. if Y_l^m denotes the spherical harmonics, then

$$\mathcal{S}_L := \Big\{ f : \mathbb{S}^2 \to \mathbb{C} : \ f = \sum_{l=0}^{L} \sum_{m=-l}^{l} a_l^m Y_l^m, \ a_l^m \in \mathbb{C} \Big\}.$$

Estimates for the p-concentration problem are of particular interest for example in geo-sciences where measurements like satellite images, or weather data are not available on the whole sphere. The Bell-Lab approach to concentration in $\mathcal{S}_L$ has numerically been applied in [20].

The maximum Nyquist density on $\mathbb{S}^2$, tailored to $\mathcal{S}_L$ is defined as

$$\rho_{\mathbb{S}^2}(\Omega, L) = \sup_{x\in\mathbb{S}^2} \frac{|\Omega \cap B_{\mathbb{S}^2}(x, t_{L,L})|}{|B_{\mathbb{S}^2}(x, t_{L,L})|}, \tag{7}$$

where the area $|\cdot|$ is measured w.r.t. the shift invariant surface measure, $t_{L,L}$ denotes the largest zero of the Legendre polynomial P_L, and $B_{\mathbb{S}^2}(x, t_{L,L})$ denotes the spherical cap with angle $\arccos t_{L,L}$ centered at $x \in \mathbb{S}^2$. Note that $t_{L,L}$ is an increasing sequence converging to 1.

In [14] estimates for the p-concentration problem are given for $1 < p < \infty$. In particular, the result in the Hilbertian case reads:

$$\lambda_2(\Omega, \mathcal{S}_L) \le A_L \cdot \rho_{\mathbb{S}^2}(\Omega, L), \tag{8}$$

where

$$A_L := (1 - t_{L,L}) \left(\int_{t_{L,L}}^{1} P_L^2(t) dt \right)^{-1}, \quad L = 1, 2, \ldots,$$

is optimal within the chosen approach. The sequence A_L is convergent with limit

$$\lim_{L\to\infty} A_L = J_1(j_{0,1})^{-2} \approx 3.71038068570948,$$

where $j_{\alpha,m}$ denotes the m-th positive zero of the Bessel function of the first kind J_α.

C. Large Sieve Principles for Gabor Spaces

Let $z = (x, \omega) \subset \mathbb{R}^2$, and define the time frequency shift π as $\pi(z)f(t) = e^{2\pi i\omega t} f(t - x)$. The short-time Fourier transform (STFT) is defined as

$$V_g f(z) = \langle f, \pi(z)g \rangle = \int_{\mathbb{R}} f(t) e^{-2\pi i \omega t} g(t - x) dt.$$

Hereafter, we restrict the choice of windows g to the class of Hermite functions h_r, and define the planar maximum Nyquist density as

$$\rho_{\mathbb{R}^2}(\Omega, R) := \sup_{z\in\mathbb{R}^2} \frac{|\Omega \cap B_{\mathbb{R}^2}(z, R)|}{|B_{\mathbb{R}^2}(z, R)|}. \tag{9}$$

The main result of [3], [4] (where it is actually proved for $1 \le p < \infty$) is the following:

Theorem 1: Let $\Omega \subset \mathbb{R}^2$ and $V_{h_r} f \in L^1(\mathbb{R}^2)$. For every $0 < R < \infty$, and every $r \in \mathbb{N}_0$,

$$\|V_{h_r} f \cdot \chi_\Omega\|_1 \le \frac{\pi R^2}{C_r(R)} \cdot \rho_{\mathbb{R}^2}(\Omega, R) \cdot \|V_{h_r} f\|_1, \tag{10}$$

with $C_r(R) = 1 - P_r(R) e^{-\pi R^2}$, and P_r a polynomial of degree $2r$ satisfying $P_r(0) = 1$, and $P_0 \equiv 1$.

The result crucially relies on the following local reproducing formula

$$V_{h_r} f(z) = \frac{1}{C_r(R)} \int_{B_{\mathbb{R}^2}(z,R)} V_{h_r} f(w) \langle \pi(w) h_r, \pi(z) h_r \rangle dw,$$

which is shown in [4] via the correspondence between the STFT with Hermite windows and polyanalytic Bargmann-Fock spaces [1].

As an illustrative application of the above theorem in the case $r = 0$, i.e. the case of Gaussian window $\varphi = h_0$, suppose that one observes only the time-frequency content of a STFT outside a region Ω, $H := P_{\Omega^c} V_\varphi f \in L^1(\mathbb{R}^2)$, and that Ω satisfies $\rho_{\mathbb{R}^2}(\Omega, R) < (1 - e^{-\pi R^2})/(2\pi R^2)$. Then:

$$V_\varphi f = \underset{V_\varphi g \in L^1(\mathbb{R}^2)}{\operatorname{argmin}} \left\| V_\varphi g \right\|_1, \ \text{ subject to } P_{\Omega^c}(V_\varphi g) = H.$$

D. The Hyperbolic Case: Bergman Spaces and Analytic Wavelets

Let $\varrho_{\mathbb{D}}$ denote the pseudohyperbolic metric in the disc $\mathbb{D}$

$$\varrho_{\mathbb{D}}(w, z) = \left| \frac{w - z}{1 - w\overline{z}} \right|,$$

and let $B_{\mathbb{D}}(z,R)$ be the pseudohyperbolic ball of center $z \in \mathbb{D}$ and radius $R < 1$ defined as $B_{\mathbb{D}}(z,R) = \{w \in \mathbb{D} : \varrho_{\mathbb{D}}(w,z) < R\}$. Moreover, we define the hyperbolic measure of a set Ω as

$$|\Omega|_{\mathbb{D}} := \int_{\Omega} (1-|z|^2)^{-2} dz.$$

The Bergman space $A^p_\alpha(\mathbb{D})$ [13] on the unit disc is defined as the space of all analytic functions F on $\mathbb{D}$ such that

$$\|F\|^p_{A^p_\alpha(\mathbb{D})} = \frac{1}{\pi}\int_{\mathbb{D}} |f(z)|^p \,(1-|z|^2)^{\alpha-2}\, dz < \infty.$$

The reproducing kernel of $A^2_\alpha(\mathbb{D})$ is given by

$$\mathcal{K}^\alpha_{\mathbb{D}}(z,w) = (1-z\overline{w})^{-\alpha}.$$

One can define a hyperbolic maximum Nyquist density as

$$\rho_{\mathbb{D}}(\Omega,R) := \sup_{z\in\mathbb{D}} \frac{|\Omega \cap B_{\mathbb{D}}(z,R)|_{\mathbb{D}}}{|B_{\mathbb{D}}(z,R)|_{\mathbb{D}}}.$$

It is then possible to obtain a hyperbolic analogue of (10).

Theorem 2: Let $\Omega \subset \mathbb{D}$ and $F \in A^1_\alpha(\mathbb{D})$. For every $R < 1$,

$$\|F \cdot \chi_\Omega\|_{A^1_\alpha(\mathbb{D})} \le 2\frac{C^0(R)}{C^\alpha(R)} \cdot \rho_{\mathbb{D}}(\Omega,R) \cdot \|F\|_{A^1_\alpha(\mathbb{D})}, \quad (11)$$

where $C^\alpha(R) = \frac{1}{\alpha-1}\left(1-(1-R^2)^{\alpha-1}\right)$.

The following local reproducing formula obtained by Seip [19, Theorem 2.6] plays a key role in the proof:

$$f(z) = \frac{1}{C^\alpha(R)} \int_{B_{\mathbb{D}}(z,R)} f(z)\mathcal{K}^\alpha_{\mathbb{D}}(z,w)(1-|w|^2)^{\alpha-2}\, dw.$$

For $p = 2$, the Bergman space $A^2_\alpha(\mathbb{D})$ is conformally equivalent to the Bergman space on the upper half-plane $A^2_\alpha(\mathbb{C}^+)$. The spaces $A^p_\alpha(\mathbb{C}^+)$ can be understood, up to a weight, as the phase space of a continuous *wavelet* transform with analyzing wavelets of the form

$$\widehat{g_\alpha}(\xi) := \frac{2^{(\alpha-1)/2}}{\Gamma(\alpha-1)^{1/2}}\, \xi^{(\alpha-1)/2} e^{-\xi} \chi_{[0,\infty)}(\xi).$$

In that case, using the conformal map between $\mathbb{D}$ and $\mathbb{C}^+$, the reproducing formula (II-D) can be moved to $A^1_\alpha(\mathbb{C}^+)$ and an equivalent estimate as (11) can be shown. Details will be given elsewhere, together with the extension to the class of wavelets which have phase space representations in polyanalytic Bergman spaces [2].

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