Frame representations via suborbits of bounded operators

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***Abstract*—The standard setup of dynamical sampling concerns frame properties of sequences of the form $\{T^n\varphi\}_{n=0}^\infty$, where T is a bounded operator on a Hilbert space $\mathcal{H}$ and $\varphi \in \mathcal{H}$. In this paper we consider two generalizations of this basic idea. We first show that the class of frames that can be represented using iterations of a bounded operator increases drastically if we allow representations using just a subfamily $\{T^{\alpha(k)}\varphi\}_{n=0}^\infty$ of $\{T^n\varphi\}_{n=0}^\infty$; indeed, any linear independent frame has such a representation for a certain bounded operator T. Furthermore, we prove a number of results relating the properties of the frame and the distribution of the powers $\{\alpha(k)\}_{k=1}^\infty$ in $\mathbb{N}$. Finally we show that also the condition of linear independency can be removed by considering approximate frame representations with an arbitrary small prescribed tolerance, in a sense to be made precise.**

I. Introduction

Dynamical sampling was introduced by Aldroubi, Davis and Kristhal in [3] in 2015 and since then it has attracted the attention of many researchers, see, e.g., [1], [2], [4], [6]. One of the key issues in dynamical sampling is the analysis of the frame properties of a sequence $\{T^n\varphi\}_{n=0}^\infty$ where T is a bounded operator on a Hilbert space $\mathcal{H}$ and $\varphi \in \mathcal{H}$. The set $\{T^n\varphi\}_{n=0}^\infty$ is called the *orbit of φ under the operator T*.

The purpose of this paper is to analyse the more general question of representation of a given frame using only a *suborbit,* i.e., a subset of the orbit of a given operator. As we will see this removes several of the constraints that appear in dynamical sampling. Finally we will consider approximate representations of a frame using subsets of the orbit of a bounded operator; this removes the remaining constraints and indeed leads to a representation that applies to arbitrary frames, at the price of an arbitrary prescribed error-margin. If we for any given application choose the error-margin smaller than the machine precision, the approximating frames will then behave exactly as good as the given frame.

In the rest of this introduction we provide a short survey of some of the standard results within dynamical sampling. In Section II we consider frame representations using suborbits, and show that this idea greatly enlarges the class of frames that can be handled. We also provide a number of results relating the overcompleteness of the given frame and the structure of the suborbit. Finally, in Section III we show that the remaining constraints can be removed by using approximate frame representations, in a sense that will be made precise.

Let us first state some conventions and basic definitions. Throughout the entire paper, $\mathcal{H}$ will denote a separable and infinite-dimensional Hilbert space. Considering a frame $\{f_k\}_{k=1}^\infty$ for $\mathcal{H}$, the synthesis operator is

$$U : \ell^2(\mathbb{N}) \to \mathcal{H},\ U\{c_k\}_{k=1}^\infty = \sum_{k=1}^\infty c_k f_k.$$

The frame operator for $\{f_k\}_{k=1}^\infty$ is $S := UU^*$ and it is invertible. The sequence $\{S^{-1}f_k\}_{k=1}^\infty$ is the *canonical dual frame* of $\{f_k\}_{k=1}^\infty$. A Bessel sequence $\{g_k\}_{k=1}^\infty$ is called a *dual frame* of $\{f_k\}_{k=1}^\infty$ if $f = \sum_{k=1}^\infty \langle f, f_k\rangle g_k$, for all $f \in \mathcal{H}$. The *excess of frame* is the maximal number of elements that can be removed yet the remaining sequence is a frame. A sequence $\{f_k\}_{k=1}^\infty$ is called a Riesz basis if it spans $\mathcal{H}$ and there exist constants $0 < A \le B$ such that

$$A\sum |c_k|^2 \le \|\sum c_k f_k\|^2 \le B\sum |c_k|^2$$

for all finite sequences $\{c_k\}$. Riesz bases are precisely the frames with zero excess. Finally, the right-shift operator on $\ell^2(\mathbb{N})$ is given by

$$\mathcal{T}(c_1, c_2, \dots) = (0, c_1, c_2, \dots).$$

In the rest of this section we present a survey of results about the class of frames $\{f_k\}_{k=1}^\infty$ that have a representation via iterated actions of a bounded operator. In particular, every Riesz basis can be represented as the orbit of a bounded operator and an overcomplete frame with finite excess never has such a representation:

Theorem I.1

(i) **[9]** *Any Riesz basis $\{f_k\}_{k=1}^\infty$ has the form $\{f_k\}_{k=1}^\infty = \{T^n f_1\}_{n=0}^\infty$ for some bounded operator $T : \mathcal{H} \to \mathcal{H}$.*

(ii) **[9], [13]** *If an overcomplete frame* $\{f_k\}_{k=1}^{\infty}$ *has a representation* $\{f_k\}_{k=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$ *with a bounded operator* T*, then* $\{f_k\}_{k=1}^{\infty}$ *has infinite excess and*

$$f_k \to 0 \text{ as } k \to \infty.$$

In [1], a particular class of overcomplete frames, the so-called Carleson frame were introduced, and it was shown that they can be represented via iterated actions of a diagonal operator; these frames were studied more in [2], [4]. It is proved in [13] that the class of frames that can represented via iterated actions of a bounded operator is indeed much larger than the class of Carleson frames but the proof does not provide a direct way of constructing such frames explicitly.

Note that Theorem I.1 puts severe restrictions on a frame $\{f_k\}_{k=1}^{\infty}$ in order to have a representation as a full orbit of a bounded operator. For example, all overcomplete Gabor frames and wavelet frames are excluded because they consist of vectors having the same norm. It was shown in [9] that the restrictions basically arise because we are asking for the representing operator T to be bounded; in fact, any linearly independent frame can be represented as an orbit of a possibly unbounded operator. We refer to [9] for details. The difficulty in representing frames using the full orbit of a bounded operator is precisely the motivation behind the current paper.

II. Frame representations using suborbits

We have seen in Theorem I.1 that several conditions are necessary for a frame $\{f_k\}_{k=1}^{\infty}$ to have a representation of the form $\{T^n\varphi\}_{n=0}^{\infty}$ for a bounded operator $T : \mathcal{H} \to \mathcal{H}$. The purpose of this section is to explore the freedom that is obtained by allowing representations of a frame that only uses certain selected powers $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ of the operator T, rather than all nonnegative powers of T.

We first show that every linearly independent frame $\{f_k\}_{k=1}^{\infty}$ can be represented in the form $\{T^{\alpha(k)}f_1\}_{k=1}^{\infty}$ where T is a bounded operator. Furthermore, we prove a number of results relating the properties of frame and the distribution of the powers $\{\alpha(k)\}_{k=1}^{\infty}$ in $\mathbb{N}$ and the distribution of the powers $\{\alpha(k)\}_{k=1}^{\infty}$ in $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. More precisely, we show that if $\{f_k\}_{k=1}^{\infty}$ has finite excess, then the sequence $\{\alpha(k)\}_{k=1}^{\infty}$ does not include any infinite subfamily consisting of consecutive numbers. We also show that if the frame $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ has finite excess, then $\alpha(k) - k \to \infty$ as $k \to \infty$.

When considering sequences $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$, we will in general assume that $\alpha(k) \neq \alpha(\ell)$ for $k \neq \ell$. Frequently, we will order the vectors $T^{\alpha(k)}\varphi$ such that

$$0 \le \alpha(1) < \alpha(2) < \cdots . \tag{II.1}$$

The following lemma proves that all sequences of the form $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ are linear independent as long as they span an infinite-dimensional subspace of $\mathcal{H}$. Note that we do not need to assume any frame properties for the sequence $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ for this particular result.

Lemma II.1 *Assume that* $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ *spans an infinite-dimensional space and that* $\alpha(k) \neq \alpha(\ell)$ *for* $k \neq \ell$*. Then* $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ *is linearly independent.*

Proof. Assume that $\alpha(k) \neq \alpha(\ell)$ for $k \neq \ell$ and that $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ is linearly dependent. Possibly after a reordering we can assume that $\alpha(k) < \alpha(k+1)$ for all $k \in \mathbb{N}$. Now, choose $N \in \mathbb{N}$ and some scalar coefficients $\{c_k\}_{k=1}^{N}$ such that $c_N \neq 0$ and $\sum_{k=1}^{N} c_k T^{\alpha(k)}\varphi = 0$. Then

$$\begin{aligned} T^{\alpha(N)}\varphi &\in \text{span}\{T^{\alpha(k)}\varphi\}_{k=1}^{N-1} \\ &\subseteq \text{span}\{T^{k}\varphi\}_{k=1}^{\alpha(N-1)} \\ &\subseteq V := \text{span}\{T^{k}\varphi\}_{k=1}^{\alpha(N)-1}, \end{aligned}$$

which implies that

$$\begin{aligned} & T^{\alpha(N)+1}\varphi \\ \in\ & \text{span}\{T^{k}\varphi\}_{k=1}^{\alpha(N)} \\ =\ & \text{span}\left(\{T^{k}\varphi\}_{k=1}^{\alpha(N)-1} \cup \{T^{\alpha(N)}\varphi\}\right) \\ =\ & V. \end{aligned}$$

Inductively this implies that $T^{\alpha(N)+\ell}\varphi \in V$ for all $\ell \in \mathbb{N}$, but this contradicts that $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ spans an infinite-dimensional space. Thus we conclude that $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ is indeed linearly independent. □

A classic result by Halperin, Kitai, Rosenthal [15] shows that the converse of Lemma II.1 holds. Indeed, any linearly independent sequence $\{v_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ has a representation using irregular powers of a certain bounded operator:

Theorem II.2 [15] *Assume that* $\mathcal{F} = \{v_k\}_{k=1}^{\infty}$ *is a countable (finite or infinite) linearly independent subset of* $\mathcal{H}$*. Then, for any* $a > 1$ *there exists an operator* $T \in B(\mathcal{H})$ *and a sequence* $\{\alpha(k)\}_{k=1}^{\infty} \subset \mathbb{N}_0$ *such that* $\{v_k\}_{k=1}^{\infty} = \{T^{\alpha(k)}v_1\}_{k=1}^{\infty}$ *and* $\|T\| = a$*.*

Theorem II.2 in particular applies to linear independent frames. Thus we obtain the next result directly from Lemma II.1 and Theorem II.2:

Corollary II.3 *Assume that* $\{f_k\}_{k=1}^{\infty}$ *is frame for* $\mathcal{H}$*. Fix any any* $a > 1$*. Then there exists an operator* $T \in B(\mathcal{H})$ *and a sequence* $\{\alpha(k)\}_{k=1}^{\infty} \subset \mathbb{N}_0$ *such that* $\{f_k\}_{k=1}^{\infty} = \{T^{\alpha(k)}f_1\}_{k=1}^{\infty}$ *and* $\|T\| = a$ *if and only if* $\{f_k\}_{k=1}^{\infty}$ *is linearly independent.*

Unfortunately, the proof of Theorem II.2 does not provide a direct approach to access the operator T or the sequence of powers $\{\alpha(k)\}_{k=1}^{\infty}$. In the rest of this section we will assume that $\{T^{\alpha(k)}f_1\}_{k=1}^{\infty}$ is a frame and discuss certain relations between the distributions of the powers $\{\alpha(k)\}_{k=1}^{\infty}$ and the excess of the frame. In the first result, stated in Proposition II.4 below, we will assume that there exists an $N \in \mathbb{N}$ such that

$$\alpha(N+\ell) = \alpha(N) + \ell, \; \forall \ell \in \mathbb{N}. \tag{II.2}$$

Note that the assumption (II.2) means that from a certain index, the vectors in $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ appear by consecutive powers of T; that is, the suborbit $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ contains all powers $T^{\ell}\varphi$ for sufficiently large values of $\ell \in \mathbb{N}$.

Proposition II.4 *Let $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ be a frame, and assume that* (II.2) *holds for some $N \in \mathbb{N}$. Then the following hold:*

(i) *If $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ has positive and finite excess, then T is unbounded.*

(ii) *If $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ is a Riesz basis, ordered such that* (II.1) *holds, and*

$$\alpha(k+1) - \alpha(k) > 1 \textit{ for some } k \in \mathbb{N},$$

then T is unbounded.

Proof. Let us for a moment consider the full orbit $\{T^k\varphi\}_{k=0}^{\infty}$ as a set. The assumption (II.2) implies that

$$\begin{aligned} & \{T^k\varphi\}_{k=0}^{\infty} \\ = \; & \{T^{\alpha(k)}\varphi\}_{k=1}^{\infty} \\ \cup \; & \{T^k\varphi\}_{k\in\{0,\dots,\alpha(N)-1\}\setminus\{\alpha(1),\dots,\alpha(N-1)\}}. \end{aligned} \tag{II.3}$$

In order to prove (i), assume now that $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ has positive and finite excess. It then follows from (II.3) that $\{T^k\varphi\}_{k=0}^{\infty}$ is a frame with finite and positive excess; hence T is unbounded by Theorem I.1. The result in (ii) follows from the same argument, using that the extra assumption in (ii) forces $\{T^k\varphi\}_{k=0}^{\infty}$ to have strictly positive excess. □

The assumption (II.2) can not be removed from Proposition II.4. In order to see this, assume that $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis. There exists $g_1 \in \mathcal{H}$ such that $\{f_k\}_{k=1}^{\infty} \cup \{g_1\}$ is linearly independent; otherwise every $f \in \mathcal{H}$ would have a representation as a finite linear combination of the vectors $\{f_k\}_{k=1}^{\infty}$, contradicting the assumption that $\mathcal{H}$ is infinite-dimensional. Proceeding inductively and assuming that we have chosen $g_1, g_2, \cdots, g_{n-1}$ for some $n \geq 2$, we can then choose g_n such that $\{f_k\}_{k=1}^{\infty} \cup \{g_1, g_2, \cdots, g_{n-1}, g_n\}$ is linearly independent, by the same argument; then $\{f_k\}_{k=1}^{\infty} \cup \{g_k\}_{k=1}^{\infty}$ is linearly independent. By Theorem II.2 there is a bounded operator T and $\varphi \in \mathcal{H}$ such that $\{f_k\}_{k=1}^{\infty} \cup \{g_k\}_{k=1}^{\infty} = \{T^k\varphi\}_{k=1}^{\infty}$. Take $\alpha(k)$ such that $\{f_k\}_{k=1}^{\infty} = \{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$. Clearly, for some $k \in \mathbb{N}$, $\alpha(k+1) - \alpha(k) > 1$. The reason that the boundedness of the operator T does not contradict the conclusion in Proposition II.4 (ii) is that the Riesz basis $\{f_k\}_{k=1}^{\infty} = \{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ does not satisfy the condition (II.2).

The result in Proposition II.4 (ii) can be rephrased as follows: if $\{T^{\alpha(k)}\varphi\}$ is a Riesz basis for a bounded operator T and $\{\alpha(k)\}_{k=1}^{\infty}$ satisfies (II.1) and (II.2), then actually

$$\{\alpha(k)\}_{k=1}^{\infty} = \{\alpha(1), \alpha(1)+1, \alpha(1)+2, \dots\}.$$

By Corollary II.3, any linearly independent frame with finite excess can be represented as a suborbit of a bounded operator. In this case, the sequence $\{\alpha(k)\}_{k=1}^{\infty}$ must be a proper subset of $\mathbb{N}$ by Theorem I.1. We will now show that the sequence $\alpha(k)$, $k \in \mathbb{N}$, has to grow with a certain speed: indeed, $\alpha(k) - k \to \infty$ as $k \to \infty$.

Corollary II.5 *Assume that $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ is an overcomplete frame with finite excess and that T is bounded. Then*

$$\alpha(k) - k \to \infty \textit{ as } k \to \infty.$$

Proof. By Proposition II.4 it is impossible that there exists $N_0 \in \mathbb{N}$ such that $\alpha(N+1) = \alpha(N) + 1$ for all $N > N_0$. Thus there exists infinitely many $N \in \mathbb{N}$ such that $\alpha(N+1) > \alpha(N) + 1$; this implies that

$$\sum_{k=1}^{\infty} [\alpha(k+1) - \alpha(k) - 1] = \infty.$$

Thus

$$\begin{aligned} & \alpha(K+1) - \alpha(1) - K \\ = \; & \sum_{k=1}^{K} [\alpha(k+1) - \alpha(k) - 1] \to \infty \end{aligned}$$

as $K \to \infty$, which leads to the result. □

III. Approximation of frames

In Section II we saw that we gain considerably flexibility by allowing a frame to be represented using a suborbit of a bounded operator rather than the whole orbit. On the other hand Lemma II.1 showed that linearly dependent frames spanning an infinite-dimensional Hilbert space can not even be represented via a suborbit. In this section we show that we can overcome this restriction by considering approximate representations of the given frame, with any prescribed tolerance in a sense to be stated in an exact way below. Indeed, at the price of considering a perturbation of a given frame $\{f_k\}_{k=1}^{\infty}$, the frame constructed in Proposition III.1 below avoids all the restrictions we know for frame representations of the form $\{f_k\}_{k=1}^{\infty} = \{T^n\varphi\}_{n=0}^{\infty}$:

- The given frame $\{f_k\}_{k=1}^{\infty}$ is not necessarily linearly independent; we can even have repeated elements.
- The given frame $\{f_k\}_{k=1}^{\infty}$ does not need to satisfy that $f_k \to 0$ as $k \to \infty$;
- The given frame $\{f_k\}_{k=1}^{\infty}$ can have finite excess.

The approximation considered in Proposition III.1 will be given in terms of a suborbit of a hypercyclic operator $T \in B(\mathcal{H})$.

Recall that an operator $T \in B(\mathcal{H})$ is called *hypercyclic* if there exists a vector $\varphi \in \mathcal{H}$ such that $\{T^n\varphi\}_{n=0}^{\infty}$ is dense in $\mathcal{H}$. The vector φ is called *hypercyclic vector* of T. Although the definition of a hypercyclic operators seems very restrictive at a first glance, the set of hypercyclic operators is actually dense in $B(\mathcal{H})$ with respect to the strong norm topology [7]. The first hypercyclic operator was constructed by Rolewicz [16]; he showed that considering the left shift operator

$$\mathcal{L} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), \quad \mathcal{L}(c_1, c_2, \dots) = (c_2, c_3, \dots)$$

and any constant $\lambda > 1$, the operator $\lambda\mathcal{L}$ is a hypercyclic operator for $\ell^2(\mathbb{N})$. Also note that if T is a hypercyclic operator with hypercyclic vector φ, then every element of the set $\{T^n\varphi \,:\, n \in \mathbb{N}\}$ is a hypercyclic vector, too; thus the set of hypercyclic vectors for T is dense in $\mathcal{H}$. For more information concerning hypercyclic operators, see e.g. [14].

Proposition III.1 *Let $T : \mathcal{H} \to \mathcal{H}$ be a hypercyclic operator, with hypercyclic vector $\varphi \in \mathcal{H}$. Given any sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ and any constant $C > 0$, there exist an increasing sequence*

$$0 \le \alpha(1) < \alpha(2) < \cdots$$

such that

$$||f_k - T^{\alpha(k)}\varphi||^2 \le C2^{-k}, \, \forall k \in \mathbb{N}. \tag{III.1}$$

Assume now additionally that $\{f_k\}_{k=1}^{\infty}$ is a frame with lower frame bound A and take any $0 < C < A$. Then the following hold:

(i) *If the sequence $\{\alpha(k)\}_{k=1}^{\infty}$ is chosen such that* (III.1) *holds, then $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with the same excess as $\{f_k\}_{k=1}^{\infty}$.*

(ii) *For any given $N \in \mathbb{N}$, the sequence $\{\alpha(k)\}_{k=1}^{\infty}$ can be chosen such that*

$$\{\alpha(k)\}_{k=1}^{\infty} \subset N\mathbb{N}_0,$$

but it is impossible that $\{\alpha(k)\}_{k=1}^{\infty} = N\mathbb{N}_0$.

Proof. The existence of a sequence $\alpha(k)$, $k \in \mathbb{N}$, satisfying (III.1) follows directly from φ being hypercyclic. Now, assuming that $\{f_k\}_{k=1}^{\infty}$ is a frame, letting A denote a lower frame bound for $\{f_k\}_{k=1}^{\infty}$, and taking $C \in]0, A[$, we obtain that $\sum_{k=1}^{\infty} ||f_k - T^{\alpha(k)}\varphi||^2 < A$; by a standard perturbation result, e.g., [8, Theorem 22.1.1], this implies that $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with the same excess as $\{f_k\}_{k=1}^{\infty}$. For the last part of the proposition, note that if T is a hypercyclic operator with respect to $\varphi \in \mathcal{H}$, then the same is the case for T^N for any $N \in \mathbb{N}$ by [5]. Thus, replacing T by T^N it follows that there is some $\{\alpha(k)\}_{k=1}^{\infty}$ for which $\{(T^N)^{\alpha(k)}\varphi\}_{k=1}^{\infty} = \{T^{\alpha(k)N}\varphi\}_{k=1}^{\infty}$ is a frame. Clearly $\{\alpha(k)\}_{k=1}^{\infty} \neq N\mathbb{N}$ because T^N is hypercyclic and therefore $\{T^{N\ell}\varphi\}_{\ell=1}^{\infty}$ is dense in $\mathcal{H}$ and hence not a frame. □

Indeed there are infinitely many choices of a sequence $\{\alpha(k)\}_{k=1}^{\infty}$ such that Proposition III.1 applies. In general the distribution of these sequences is not known exactly. In a forthcoming paper [11], the authors will provide explicitly given applicable sequences $\{\alpha(k)\}_{k=1}^{\infty}$ for certain prescribed operators T.

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