Phase Retrieval for Wide Band Signals

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***Abstract*—This study investigates the phase retrieval problem for wideband signals. More precisely, we solve the following problem: given $f \in L^2(\mathbb{R})$ with Fourier transform in $L^2(\mathbb{R}, e^{2c|x|}\,\mathbf{d}x)$ we determine all functions $g \in L^2(\mathbb{R})$ with Fourier transform in $L^2(\mathbb{R}, e^{2c|x|}\,\mathbf{d}x)$, such that $|f(x)| = |g(x)|$ for all $x \in \mathbb{R}$. To do so, we translate the problem into a phase retrieval problem for functions in Hardy spaces on the disc and use the inner-outer factorization.**

***Index Terms*—Hardy spaces, phase retrieval**

I. Introduction

The phase retrieval problem refers to the recovery of the phase of a function f using given data on its magnitude $|f|$ and an a priori on f. This class of problems is widely studied because of its various physical applications, such as in astronomy [8], lens design [9], x-ray crystallography [20], inverse scattering [22], and optics [23]. Other physical examples are given in the survey articles of Klibanov et. al. [15] and the book [11] which provide a rather comprehensive overview of the mathematical literature on the phase retreival problem at the time of their publication.

Recently, the phase retrieval problem has met a burst of interest in the mathematical community due to the appearance of new algorithms starting with the work of Candès et. al. [7] and of Waldspurger et. al. [25]. This led to a bulk of work on the phase retrieval problem in the discrete (finite-dimensional) setting which is also the right setting for numerical algorithms.

However, most physical problems take naturally place in the continuous setting. For instance, the phase retrieval problem has been solved for 1-D band-limited functions in [1], [2], [26], for functions in the Hardy space of the disc with a technical restriction [5], for samples of real-valued band-limited functions [24] and Sobolev-functions in [10], (see [3] for more on this problem, in particular the stability of the problem and further references). Our aim here is to investigate the phase retrieval problem for wide-band functions, namely functions with mildly decreasing Fourier transforms.

Before outlining our results, let us give a quick overview of the band-limited and narrow band cases. In the mid-1950's, Akutowicz [1], [2] (and independently also in 1963, Walther [26]) solved the phase retrieval problem in the class of compactly supported functions: given a band-limited function $f \in L^2(\mathbb{R})$ (*i.e.* a function with compactly supported Fourier transform), find all band-limited functions $g \in L^2(\mathbb{R})$ such that

$$|f(x)| = |g(x)| \qquad \text{for every } x \in \mathbb{R}. \tag{1}$$

Let us sketch the solution of this problem. The first step consists in using the Paley-Wiener Theorem which states that f and g extend into holomorphic functions in the plane that are of exponential type, namely they grow like $e^{a|z|}$. The second step consists in observing that (1) is then equivalent to

$$f(z)\overline{f(\bar{z})} = g(z)\overline{g(\bar{z})} \qquad \text{for every } z \in \mathbb{C}. \tag{2}$$

Indeed, (2) is a reformulation of (1) when z is *real* and is an equality between two holomorphic functions so that it is valid for all $z \in \mathbb{C}$. The final step consists in using the Hadamard Factorization Theorem which states that holomorphic functions of exponential type are essentially characterized by their zeros. But now, as observed by Akutowicz and Walther, (2) implies that each zero of g is either a zero of f or a complex conjugate of a such a zero. To reconstruct g, one thus changes arbitrarily many zeroes of f into their complex zeros in the Hadamard factorization of g and this is called *zero-flipping*.

This proof has then been extended by McDonald [18] to functions that have Fourier tranforms with very fast decrease at infinity, for instance, Gaussian decrease. Indeed, if $|\widehat{f}(\xi)|, |\widehat{g}(\xi)| \lesssim e^{-a|\xi|^2}$, $a > 0$, then f, g extend to holomorphic functions of type 2 so that Hadamard factorization is still available. Thus, the solutions are again essentially given by zero-flipping.

This method of proof actually extends to functions that satisfy a decay condition of the form $|\widehat{f}(\xi)|, |\widehat{g}(\xi)| \lesssim e^{-a|\xi|^\alpha}$, $a > 0$ and $\alpha > 1$ but breaks down at $\alpha = 1$. It is precisely the aim of this work to investigate the phase retrieval problem for functions that have Fourier transform with exponential decay: $|\widehat{f}(\xi)|, |\widehat{g}(\xi)| \lesssim e^{-a|\xi|}$. This class of functions is sometimes called wide-band signals in the engineering community. In this case, the functions f and g only extend holomorphically to an horizontal strip $\mathcal{S}_a = \{z \in \mathbb{C} : |\Im(z)| < a\}$ in the complex plane so that (2) is only valid for $z \in \mathcal{S}_a$. Further, Hadamard factorization is no longer available. To overcome these difficulties, we first reduce the problem to the Hardy space on the disc via a conformal transform. We then exploit the inner-outer-Blaschke factorization in the Hardy space on the disc and finally, a new conformal transform allows us to fully solve the wide-band phase retrieval problem.

The solution is now more evolved than the band-limited case as, in addition to zero-flipping, the inner and the outer part also contribute to the solution set. In future work, we will investigate how various means that allow to reduce the solution set in the band-limited case may also reduce the solution set here.

This work is organized as follows. Section II is a quick review of definitions and results on Hardy spaces. Section III is devoted to the sketch of the solution of the phase retrieval problem in the wide-band case. We conclude this note with an overview of potential results that might reduce the set of solutions.

II. Preliminaries

A. Notation

For a domain $\Omega \subset \mathbb{C}$, $\mathrm{Hol}(\Omega)$ is the set of holomorphic functions on Ω. For $F \in \mathrm{Hol}(\Omega)$ we denote by $Z(F)$ the set of zeros of F, counted with multiplicity. If Ω is stable by complex conjugation and $F \in \mathrm{Hol}(\Omega)$, we denote by F^* the function in $\mathrm{Hol}(\Omega)$ defined by $F^*(z) = \overline{F(\bar{z})}$. We also denote the conjugation function by C, where $C(z) = \bar{z}$ for all $z \in \mathbb{C}$.

We also consider the unit disc $\mathbb{D}$ defined as $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and its boundary $\mathbb{T}$ defined by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. For $c > 0$, let $\mathcal{S}_c$ be the strip defined as $\mathcal{S}_c := \{z \in \mathbb{C} : |\mathrm{Im} z| < c\}$, and $\mathcal{S} := \mathcal{S}_1$.

B. Hardy spaces on the Disc

Recall that the Hardy spaces on the disc $\mathbb{D}$ are defined as

$$H^2(\mathbb{D}) = \left\{ F \in \mathrm{Hol}(\mathbb{D}) : \sup_{0 \le r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{i\theta})|^2 \,\mathrm{d}\theta < \infty \right\},$$

and

$$H^\infty(\mathbb{D}) = \left\{ F \in \mathrm{Hol}(\mathbb{D}) : \sup_{w \in \mathbb{D}} |F(w)| < \infty \right\}.$$

We will need the following key facts on Hardy spaces. First, every $F \in H^2(\mathbb{D})$ admits a radial limit $F(e^{i\theta}) = \lim_{r \to 1} F(re^{i\theta})$ for almost every $e^{i\theta} \in \mathbb{T}$ (see e.g. [17, Lemma 3.10]) with $F \in L^2(\mathbb{T})$, $\log |F| \in L^1(\mathbb{T})$ and such that the Fourier coefficients satisfy $\widehat{F}(n) = 0$ for $n = -1, -2, \ldots$. Furthermore [17, Section 7.6], every function $F \in H^2(\mathbb{D})$ can be uniquely decomposed as

$$F = e^{i\gamma} B_F S_F O_F$$

where $e^{i\gamma} \in \mathbb{T}$, B_F is the Blaschke product formed by the zeros of F, S_F is a singular inner function, and O_F is the outer part of F. Recall that the Blaschke product is defined for all $w \in \mathbb{D}$ as

$$B_F(w) = \prod_{\alpha \in Z(F)} b_\alpha(w), \tag{3}$$

where $Z(F)$ is the zero set of F repeated according to multiplicity and

$$b_\alpha(w) = \begin{cases} w & \text{if } \alpha = 0 \\ \dfrac{\alpha}{|\alpha|} \dfrac{\alpha - w}{1 - \bar{\alpha} w} & \text{if } \alpha \neq 0 \end{cases}.$$

The singular part is given by

$$S_F(w) = \exp\left(\int_{\mathbb{T}} \frac{w + e^{i\theta}}{w - e^{i\theta}} \,\mathrm{d}\nu_F\left(e^{i\theta}\right) \right), \tag{4}$$

where ν_F is a finite positive singular measure (with respect to the Lebesgue measure). Finally, the outer part is determined by the modulus of the radial limit of F

$$O_F(w) = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{w + e^{i\theta}}{w - e^{i\theta}} \log\left|F\left(e^{i\theta}\right)\right| \,\mathrm{d}\theta \right). \tag{5}$$

C. Hardy Spaces on the Strip

There are essentially two ways of defining the Hardy space on the strip $\mathcal{S}$. On one hand we shall consider the following Hardy spaces defined

$$H^2(\mathcal{S}) = \left\{ f \in \mathrm{Hol}(\mathcal{S}) : f \circ \phi^{-1} \in H^2(\mathbb{D}) \right\},$$

where $\phi : \mathcal{S} \longrightarrow \mathbb{D}$ is a bijective conformal mapping defined by

$$\phi(z) := \tanh\left(\frac{\pi}{4} z\right)$$

and $||f||_{H^2(\mathcal{S})} = ||f \circ \phi^{-1}||_{H^2(\mathbb{D})}$. It can then be shown [4, Theorem 2.2] that $H^2(\mathcal{S}) = H^2_W(\mathcal{S})$ isometrically where

$$H^2_W(\mathcal{S}) = \left\{ f \in \mathrm{Hol}(\mathcal{S}) : \sup_{|y|<1} \int_{\mathbb{R}} \frac{|f(t+iy)|^2}{|W(t+iy)|} \,\mathrm{d}t < \infty \right\},$$

and $W(z) = \dfrac{1}{4\cosh^2(\frac{\pi}{4} z)} = \pi \phi'(z)$.

Now this last space can be identified with the natural analogue of the Hardy space on the disc:

$$H^2_\tau(\mathcal{S}) = \left\{ f \in \mathrm{Hol}(\mathcal{S}) : \sup_{|y|<1} \int_{\mathbb{R}} |f(t+iy)|^2 \,\mathrm{d}t < \infty \right\}.$$

More precisely $f \in H^2_\tau(\mathcal{S})$ if and only if $W^{1/2} f \in H^2_W(\mathcal{S})$.

In summary,

Theorem II.1 ([4], Equation (2.1), Theorem 2.1)**.** *For f a function on $\mathcal{S}$, define the function $F = (W^{1/2} f) \circ \phi^{-1}$ on $\mathbb{D}$, or conversely to F on $\mathbb{D}$ we associate $f = W^{-1/2} F \circ \phi$. Then $f \in H^2_\tau(\mathcal{S})$ if and only if $F \in H^2(\mathbb{D})$.*

Finally, it is a direct consequence of Plancherel's theorem that $H^2_\tau(\mathcal{S})$ is the space of Fourier transforms of functions in $L^2(\mathbb{R}, e^{2|\xi|}\mathrm{d}\xi)$:

Theorem II.2 (Paley-Wiener on the Strip)**.** *We have $f \in H^2_\tau(\mathcal{S})$ if and only if $\widehat{f} \in L^2(\mathbb{R}, e^{2|\xi|}\mathrm{d}\xi)$.*

III. Phase Retrieval in $H^2(\mathcal{S})$

A. Reduction of the Problem

In this section, we consider $f, g \in L^2(\mathbb{R})$ with $|\widehat{f}(\xi)|, |\widehat{g}(\xi)| \lesssim e^{-c|\xi|}$, i.e. $\widehat{f}, \widehat{g} \in L^2(\mathbb{R}, e^{2c|\xi|}\mathrm{d}\xi)$, such that $|f(x)| = |g(x)|$, for every $x \in \mathbb{R}$. Our goal is to determine, for a given f, all possible g's. To do so, let us write $f_c(x) = f(cx)$ and $g_c(x) = g(cx)$ so that $f_c, g_c \in L^2(\mathbb{R})$ with $\widehat{f_c}, \widehat{g_c} \in L^2(\mathbb{R}, e^{2|\xi|}\mathrm{d}\xi)$ and $|f_c(x)| = |g_c(x)|$ for every $x \in \mathbb{R}$ so that it is enough to consider the case $c = 1$. Then, according to Theorem II.2, f and g extend holomorphically to $\mathcal{S}$ and $|f(x)| = |g(x)|$ for every $x \in \mathbb{R}$ can be written as

$$f(x)\overline{f(\bar{x})} = g(x)\overline{g(\bar{x})} \quad \text{for every } x \in \mathbb{R}. \tag{6}$$

But now, (6) is an equality between two holomorphic functions on $\mathbb{R}$ so that it is valid also for all $x \in \mathcal{S}$. In other words, we are now trying to solve the following problem: given $f \in H^2_\tau(\mathcal{S})$, find all $g \in H^2_\tau(\mathcal{S})$ such that

$$f(z) f^*(z) = g(z) g^*(z) \quad \text{for every } z \in \mathcal{S}. \tag{7}$$

It turns out that this problem is easier to solve when transfering the problem to the disc. Multiplying by $W^{1/2}(z)$ and $\overline{W^{1/2}(\bar{z})}$ to both sides of (7), we obtain

$$(W^{1/2}f)(z)\overline{(W^{1/2}f)(\bar{z})} = (W^{1/2}g)(z)\overline{(W^{1/2}g)(\bar{z})}$$

for all $z \in \mathcal{S}$. By Theorem II.1, the functions $F = W^{1/2}f \circ \phi^{-1}$ and $G = W^{1/2}g \circ \phi^{-1}$ are in $H^2(\mathbb{D})$. Hence, by applying the substitution $z = \phi^{-1}(w)$ and $\bar{z} = \phi^{-1}(\bar{w})$ to the previous equation, we get

$$F(w)F^*(w) = G(w)G^*(w) quad \text{for every } w \in \mathbb{D}. \quad (8)$$

Therefore, we have translated the equality on the strip to an equivalent equality on the disc. Finally, we are now trying to solve the following problem on the disc: given $F \in H^2(\mathbb{D})$, find all $G \in H^2(\mathbb{D})$ such that (8) holds for all $w \in \mathbb{D}$. Note that (8) is equivalent to $|F(w)|^2 = |G(w)|^2$ for $w \in (-1, 1)$.

B. Reduction to the Disc

Let $F \in H^2(\mathbb{D})$ and write $F = B_F S_F O_F$ with B_F, S_F, O_F given in equations (3), (4) and (5), respectively. The factorization of F^* is given by

$$F^* = e^{i\lambda} B_{F^*} S_{F^*} O_{F^*} = e^{i\lambda} B_F^* S_F^* O_F^*.$$

Since the factorization in $H^2(\mathbb{D})$ is unique, we have $B_{F^*} = B_F^*$, $S_{F^*} = S_F^*$, and $O_{F^*} = O_F^*$. Thus, it can be shown that the Blaschke product in F^* is associated with the set $\overline{Z(F)}$, the singular part of F^* is given by the singular inner function associated to the pullback measure $C_*\nu_F$, and the outer part of F^* is determined by the modulus of the radial limit of F^*. Using these facts, we have the following characterization of the solutions of the equivalent problem on the disc.

Theorem III.1. *Let F be in $H^2(\mathbb{D})$ and write $F = e^{i\gamma} B_F S_F O_F$ be its inner-outer decomposition with $\gamma \in \mathbb{R}$ and B_F, S_F, O_F given by* (3), (4), (5) *respectively.*

Then $G \in H^2(\mathbb{D})$ is such that $|G(x)| = |F(x)|$ for every $x \in (-1, 1)$ if and only if the inner-outer decomposition of G, $G = B_G S_G O_G$ is, up to the multiplication by a unimodular constant, where

1) *B_G is the Blaschke product associated with the set $A \cup (\overline{Z(F) \backslash A})$ for some $A \subset Z(F)$;*
2) *S_G is the singular inner function associated with the positive singular measure $\nu_G = \nu_F + \rho$, where ρ is an odd real singular measure ($C_*\rho = -\rho$) such that ν_G is still a postive measure.*
3) *there exists a function U that is holomorphic in the disc and admits a boundary value on $\mathbb{T}$ that satisfies $|U(e^{i\theta})U(e^{-i\theta})| = 1$ almost everywhere on $\mathbb{T}$ such that the outer part O_G of G is $O_G = UO_F$.*

Remark III.2. *We write $\rho = \rho_+ - \rho_-$, where ρ_+ is the positive part while ρ_- is the negative part. Note that the positive part and the negative part have disjoint supports. Since ρ is an odd measure, $C_*\rho = -\rho$, thus $\rho_- = C_*\rho_+$. This implies that if $E = supp\rho_+$, then $E \cap \overline{E} = \emptyset$ and Finally, for ν_G to be positive, we need $\rho_- = C_*\rho_+ \leq \nu_F$, or equivalently, $\rho_+ \leq C_*\nu_F$. This gives a constructive description of the measures appearing in Theorem III.1(2).*

The regularity of the functions U appearing in Theorem III.1(3) depends on F. Obviously, if $U \in H^\infty(\mathbb{D})$, $\log|U| \in L^1(\mathbb{T})$, $|U(e^{i\theta})U(e^{-i\theta})| = 1$ a.e., then $O_G = UO_F$ is an outer part of an H^2 function and thus gives rise to a solution of the phase retrieval solution. While it is likely that this is the best condition one can impose on U to always give rise to a solution, for some F's, other U's may work. For instance, when $F = 1$, one may take any outer $U \in H^2(\mathbb{D})$, such that $|U(e^{i\theta})U(e^{-i\theta})| = 1$.

We can actually identify the solutions of our phase retrieval problem on the disc in terms of a factorization. Let us consider an analog of the result of McDonald [18, Proposition 1].

Corollary III.3. *Let $F, G \in H^2(\mathbb{D})$. Then $|F| = |G|$ on $(-1, 1)$ if and only if there exist $u, v \in \mathrm{Hol}(\mathbb{D})$ such that $F = uv$ and $G = uv^*$.*

This result can be shown by further decomposing the Blaschke products, the singular inner functions, and the outer functions on the inner-outer factorizations of F and G given in Theorem III.1.

C. Back to the Strip

For $F \in H^2(\mathbb{D})$ and $z \in \mathcal{S}$, we have $F(\phi(z)) = W^{1/2}(z)f(z)$ and equivalently, the unique inner-outer factorization for $f \in H^2_\tau(\mathcal{S})$ is given by

$$f(z) = e^{i\gamma} W(z)^{-1/2} B_F(\phi(z)) S_F(\phi(z)) O_F(\phi(z))$$

for all $z \in \mathcal{S}$ and for some $\gamma \in \mathbb{R}$. Note that this is well-defined on $\mathcal{S}$ since $W(z) = \pi\phi'(z) \neq 0$ for any $z \in \mathcal{S}$. The Blaschke product B_f is given by

$$B_f(z) = B_F(\phi(z)) = \prod_{\beta \in Z(f)} b_{\phi(\beta)}(\phi(z)), \quad (9)$$

while the singular inner function S_f is given by

$$\begin{aligned} S_f(z) = S_F(\phi(z)) &= \exp\left(\int_{\mathbb{T}} \frac{\phi(z) + e^{i\theta}}{\phi(z) - e^{i\theta}} \,\mathrm{d}\nu_F\left(e^{i\theta}\right) \right) \\ &= \exp\left(\int_{\partial\mathcal{S}} \frac{\phi(z) + \phi(\zeta)}{\phi(z) - \phi(\zeta)} \,\mathrm{d}\mu_f\left(\zeta\right) \right) \end{aligned} \quad (10)$$

for all $z \in \mathcal{S}$, where $\mu_f = {\phi^{-1}}_* \nu_F$ is the pullback measure of ν_F on $\partial\mathcal{S}$. For the the outer part, we need to split the integral since the upper and the lower boundaries of $\mathcal{S}$ are mapped to the upper and the lower halves of $\mathbb{T}$ given by $\mathbb{T}^+ = \{z \in T : \Im z > 0\}$ and $\mathbb{T}^- = z \in T : \Im z < 0\}$, respectively. By applying the substitutions $e^{i\theta} = \phi(x+i)$ on $\mathbb{T}^+$ and $e^{i\theta} = \phi(x-i)$ on $\mathbb{T}^-$, we get

$$\begin{aligned} &O_f(z) = O_F(\phi(z)) \\ &= \exp\Bigg(\frac{-1}{2\pi i} \int_{\mathbb{R}} \frac{\phi(z) + \phi(x+i)}{\phi(z) - \phi(x+i)} \frac{\phi'(x+i)}{\phi(x+i)} \log|(W^{1/2}f)(x+i)| \,\mathrm{d}x \\ &\quad + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\phi(z) + \phi(x-i)}{\phi(z) - \phi(x-i)} \frac{\phi'(x-i)}{\phi(x-i)} \log|(W^{1/2}f)(x-i)| \,\mathrm{d}x \Bigg) \end{aligned} \quad (11)$$

for all $z \in \mathcal{S}$. Using equations (9), (10) and (11), we translate Theorem III.1 to functions on $H^2_\tau(\mathcal{S})$. Finally, by using Theorem II.2, we go back to the initial setting of the problem.

IV. Conclusion

In this paper, we have outlined the solution of the following phase retreival problem: given $f \in L^2(\mathbb{R})$ with $\widehat{f} \in L^2(\mathbb{R}, e^{2c|\xi|}\mathrm{d}\xi)$ find all $g \in L^2(\mathbb{R})$ with $\widehat{g} \in L^2(\mathbb{R}, e^{2c|\xi|}\mathrm{d}\xi)$ such that $|g(x)| = |f(x)|$, for all $x \in \mathbb{R}$. We have shown that this problem is easier to solve by translating the initial problem to $H^2(\mathbb{D})$ with the help of a conformal mapping. With this equivalent problem, we have shown that the solution can be characterized by the components of the inner-outer factorization in $H^2(\mathbb{D})$. Each term in this factorization: the inner part, the outer part and the Blaschke product, contribute to the solution set. The contribution of the Blaschke part corresponds to the zero-flipping of the band-limited case. We have also shown that the solution can be characterized as a product of two functions in $\mathrm{Hol}(\mathbb{D})$.

Several extensions to this problem can be considered. First of all, one would like to only assume that $|f| = |g|$ on a discrete subset of $\mathbb{R}$. This requires translating uniqueness results for functions in the Hardy of the disc into uniqueness results for functions in the Hardy space of the strip.

Further, the set of solutions is much larger than in the band-limited case. One question one may then ask is to determine to which extend additional constraints or additional (phaseless) measurements may lead to uniqueness or at least to a reduction of the set of solutions.

We have several examples in mind:

(1) Add the condition $|Df| = |Dg|$, where D is a difference operator. In the band-limited case, this has been done by McDonald [18] and the proof should also apply here.

(2) The Pauli problem: Pauli asked whether $|g| = |f|$ and $|\widehat{g}| = |\widehat{f}|$ implies $g = cf$ for some unimodular constant. The first author [13] and independently Ismagilov [12] have shown that the solution set may be arbitrarily large. However, if one adds the condition that $\widehat{f}, \widehat{g}$ should have bounded support then one can check that the set of solutions constructed in these papers has to be finite. Is it possible to adapt the construction to obtain an f with $\hat{f} \in L^2(\mathbb{R}, e^{2c|x|}\,\mathrm{d}x)$ for which Pauli's problem has an uncountable set of solutions?

(3) In the spirit of what was done by Boche et. al. [5], one may require that $|f(z)| = |g(z)|$ for z in some curve inside $\mathcal{S}_c$. Similar ides can also be found in [14] and [6].

(4) Klibanov et. al. [15] considered the following problem: $|g| = |f|$ and $|g+h| = |f+h|$ where h is a fixed reference signal. Under certain conditions this leads to only two solutions. Is this the case here as well ?

Finally, our original motivation stems from [16] where the following phase retrieval problem occurs: determine all f in a Sobolev space H^s such that $(\widehat{f})^* = |\widehat{f}|$ where $*$ means symmetric decreasing rearrangement. This leads naturally to the question of extending our results to radial higher dimensional functions.

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