Phase retrieval from local correlation measurements with fixed shift length

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***Abstract*—Driven by ptychography, we consider an extension of the phase retrieval problem from local correlation measurements with shifts of length one [1], [2], [3] to any fixed shift length. We provide an algorithm and recovery guarantees for the extended model.**

I. INTRODUCTION

A. Phase Retrieval

In ptychography [9], an object $x_0 \in \mathbb{C}^d$ is illuminated by a localized (masked) X-ray beam and the resulting spectrograms of the far field diffraction patterns are observed. A discretized measurement model for this process is given by

$$(y_\ell)_j = \left| \sum_{n=1}^{d} \tilde{m}_n (S_\ell x_0)_n e^{-\frac{2\pi i (j-1)(n-1)}{d}} \right|^2 + \eta_{\ell,j}, \qquad \text{(I.1)}$$

for $(\ell, j) \in \mathcal{I} \subseteq [d]_0 \times [d]$, where $[n] := \{1, 2, \ldots, n\}$, $[n]_0 := \{0, 1, \ldots, n-1\}$, vector $\tilde{m} \in \mathbb{C}^d$ is a mask, $\eta_{\ell,j}$ is noise and S_ℓ denotes the circular shift operators $S_\ell : \mathbb{C}^d \mapsto \mathbb{C}^d$ $(S_\ell x)_k = x_{\ell+k-1 \bmod d+1}$ and all indexes are considered modulo d. Setting $(m_j)_n := (\tilde{m}_n)^* e^{\frac{2\pi i (j-1)(n-1)}{d}}$ with b^* being complex conjugate of b, and changing the summation index, yields measurements of the form

$$(y_\ell)_j = |\langle S_\ell^* m_j, x_0 \rangle|^2 + \eta_{\ell,j}, \ (\ell, j) \in \mathcal{I}, \qquad \text{(I.2)}$$

where S_ℓ^* is the adjoint operator of S_ℓ. Further, by defining $a_{\ell,j} := S_\ell^* m_j$, the measurement model takes the form

$$y_{\ell,j} = |\langle a_{\ell,j}, x_0 \rangle|^2 + \eta_{\ell,j}, \ (\ell, j) \in \mathcal{I}.$$

The problem of retrieving x_0 from such measurements is commonly referred to as a *phase retrieval problem*. Besides ptychography, such problems arise naturally in optics [10], [11] and crystallography [12], [13]. Many algorithms have been proposed for the phase restoration, among them Gerchberg–Saxton's alternating projections algorithm [4], [5], Wirtinger flow [6], the Hybrid Input Output algorithm [14], PhaseLift [7], [8] and Block Phase Retrieval (BPR) [1], [2], [3]. The latter algorithm was designed specifically for ptychographical applications, which we will also study in this paper. We analyze a version of BPR adapted to subsampled measurements, i.e., not every shift is considered. Such measurements are preferred in practice due to technical difficulties connected with the implementation of single pixel shifts.

B. Local correlation measurements with fixed shift length

As a step towards understanding measurements of the form (I.1), we construct a family of masks $\{m_j\}_{j \in [K]}$ for some $K \in \mathbb{N}$ such that measurements of the form (I.2) for some $\mathcal{I} \subseteq [d]_0 \times [K]$ allow for recovery of x_0 and the masks are localized in the sense that they are supported in $\delta < d$ consecutive entries. W.l.o.g. we assume that the masks are appropriately shifted, so that $\operatorname{supp} m_j \subseteq [\delta]$ for all $j = 1, \ldots, K$. Such measurements are commonly referred to as *local correlation measurements* [1], [2], [3].

For the case, when $\mathcal{I} = [d]_0 \times [K]$, [1], [2], [3] provided algorithms and recovery guarantees. In this paper, we consider the measurement process with regular shifts of length $p \leq \delta$. We assume that p is divisor of d and define $d' := d/p$. Then, the measurements are given by

$$(y_\ell)_j = |\langle S_{\ell p}^* m_j, x_0 \rangle|^2 + \eta_{\ell,j}. \ j \in [K], \ell \in [d']_0, \qquad \text{(I.3)}$$

Notice, that the case of $p = 1$ directly corresponds to the setting in [1], [2], [3], thus all the results are applicable. The case $p > 1$ was first investigated in [3], where the author establishes performance guarantees for a variant of Algorithm 1 below for certain random masks. In this paper we generalize these results to certain classes of deterministic masks similar to those analyzed in [1], [2] for the scenario without subsampling.

In addition to p being a divisor of d, we require p to be a divisor of δ and will denote $\delta' := \delta/p$. This is not a strong assumption as common dimensions of images and mask supports are powers of 2.

C. Recovery Algorithm

To motivate the algorithm under consideration, we now review the ideas that lead to Algorithm 8 in [3]. We can rewrite the square of amplitude in order to introduce linearity.

$$|\langle x_0, S_{\ell p}^* m_j\rangle|^2 = \langle x_0 x_0^*, S_{\ell p}^* m_j m_j^* S_{\ell p}\rangle_F,\ j \in [K], \ell \in [d']_0,$$

where $\langle \cdot, \cdot\rangle_F$ denotes the Frobenius inner product in $\mathbb{C}^{d\times d}$ defined as $\langle A, B\rangle_F := \operatorname{tr}(A^*B)$. For a matrix $A \in \operatorname{span}\{S_{\ell p}^* m_j m_j^* S_{\ell p}\}_{\ell\in[d']_0, j\in[K]}$, we have $A_{ij} = 0$ whenever it is not possible to find $z \in [d']$ such that $(i, j) \in [zp+1, zp+\delta]^2$. Therefore, we can introduce an orthogonal projection operator $T_\delta^p : \mathbb{C}^{d\times d} \mapsto \mathbb{C}^{d\times d}$ as follows:

$$(T_\delta^p(A))_{i,j} = \begin{cases} A_{i,j}, & \text{if } \exists z \in [d'] \text{ such that} \\ & (i, j) \in [zp + 1, zp + \delta]^2 \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\operatorname{span}\{S_{\ell p}^* m_j m_j^* S_{\ell p}\} \subseteq T_\delta^p(\mathbb{C}^{d\times d})$ and we obtain, that for all $j \in [K], \ell \in [d']_0$ it holds that

$$\langle x_0 x_0^*, S_{\ell p}^* m_j m_j^* S_{\ell p}\rangle_F = \langle T_\delta^p(x_0 x_0^*), S_{\ell p}^* m_j m_j^* S_{\ell p}\rangle_F.$$

As a result, we can set $D := K\cdot d'$ and define the measurement operator $\mathcal{A} : \mathbb{C}^{d\times d} \mapsto \mathbb{C}^D$ as

$$(\mathcal{A}(X))_{\ell,j} = \langle X, S_{\ell p}^* m_j m_j^* S_{\ell p}\rangle_F, \quad j \in [K], \ell \in [d']_0.$$

If with a suitable choice of K and measurement masks $m_j, j \in [K]$, we can achieve the invertibility of the restricted operator $\mathcal{A}\big|_{T_\delta^p(\mathcal{H}^d)}$, where $\mathcal{H}^d$ denotes the set of Hermitian $d \times d$ matrices, we can recover $X_0 = T_\delta^p(x_0 x_0^*)$ from the measurements. Then, what is left is to recover x_0 from X_0. Amplitudes can be recovered from the diagonal of X_0 and the phases can be obtained by eigenvector-based angular synchronization [1], [2], [3]. As a result we obtain the following algorithm, which is a variant of Algorithms 3 and 8 from [3].

Algorithm 1. Fast Phase Retrieval from Local Correlation Measurements with Fixed Shift Length

Input: vector of measurement results $y \in \mathbb{C}^D$, parameters $1 \le \delta \le d$ and $1 \le p \le \delta$.
Output: $x \in \mathbb{C}^d$ with $x \approx e^{-i\theta} x_0$ for some $\theta \in [0, 2\pi]$.

1. Compute matrix $X = \left[\mathcal{A}\big|_{T_\delta^p(\mathcal{H}^d)}\right]^{-1} y \in T_\delta^p\left(\mathcal{H}^d\right)$,
2. Form $\tilde{X} \in T_\delta^p\left(\mathcal{H}^d\right)$, by normalizing the non-zero entries of X, that is $\tilde{X}_{i,j} = X_{i,j}/|X_{i,j}|$, when $|X_{i,j}| \neq 0$ and 0 otherwise.
3. Compute the top eigenvector $u \in \mathbb{C}^d$ of $\tilde{X}$ and set $\tilde{x} = \operatorname{sgn}(u) = (\operatorname{sgn}(u_1), \ldots, \operatorname{sgn}(u_d))$.
4. Set $x_j = \sqrt{X_{j,j}}(\tilde{x}_j)$ for all $j \in [d]$ to form $x \in \mathbb{C}^d$.

One can see that by taking $p = 1$, we would get precisely the Algorithm 1 from [1] and its modification with the small change in step 1 for $p > 1$.

II. Results

A. Recovery via hidden unit length model

The existence of the feasible measurement masks is directly follows from existing recovery guarantees for the case of $p = 1$. Indeed, consider masks of the form $m_{j,z} = S_z \tilde{m}_j$, where $z \in [p]_0$ and $\{\tilde{m}_j\}_{j\in\tilde{K}}$ is a set of masks to be determined later. By inserting $m_{j,z}$ into (I.3) we obtain

$$|\langle S_{\ell p}^* m_{j,z}, x_0\rangle| = |\langle S_{\ell p}^* S_z \tilde{m}_j, x_0\rangle| = |\langle S_t^* \tilde{m}_j, x_0\rangle|$$

for $t \in [d]_0$. As a result, we returned to the unit shift model and all recovery results are available. What is left is to select suitable $\tilde{m}_j$ satisfying $\operatorname{supp} S_z \tilde{m}_j = \operatorname{supp} m_{j,z} \subset [\delta]$. Extreme cases $z = 0$ and $z = p-1$ imply that $\operatorname{supp} \tilde{m}_j \subset [\delta-p+1]$ and $\{\tilde{m}_j\}_{j\in\tilde{K}}$ can be selected from a suitable mask constructions for unit shift model with support size $\delta - p + 1$. Then, application of the Theorem 1 from [1] grants us

Theorem II.1. *Let $\delta \ge 2 + p$ and $4(\delta - p) \le d$. Set $\beta := \delta - p + 1$. Let $\{\tilde{m}_j\}_{j=1}^{K_\beta}$ be a suitable set of masks for the unit shift length model with support size β. Then, for the p shift length model with support size δ and masks $m_{j,z} = S_z \tilde{m}_j, j \in [K_\beta], z \in [p]_0$ it holds, that the estimate x produced in Algorithm 1 with input parameters β and 1 satisfies*

$$\min_{\theta\in[0,2\pi]} \left\|x_0 - e^{i\theta} x\right\|_2 \le$$

$$C \left[\frac{\|x_0\|_\infty}{(x_0)_{min}^2}\right] \left[\frac{d}{\delta - p + 1}\right]^2 \sigma_{min}^{-1} \|\mathbf{n}\|_2 + C d^{\frac{1}{4}} \sqrt{\sigma_{min}^{-1} \|\mathbf{n}\|_2},$$

where $\sigma_{min} > 0$ is the minimal singular value of the operator $\mathcal{A}\big|_{T_\beta^1(\mathcal{H}^d)}$, $\mathbf{n}$ is a noise vector, $C > 0$ is an absolute universal constant and $(x_0)_{min} := \min_j |(x_0)_j|$.

B. General recovery guarantees

As argued in the previous paragraph, the existence of suitable masks is relatively straightforward, but also does not add much insights. Hence for better understanding it is required to understand larger families of masks. Thus, one needs a more general statement for bounding the reconstruction error of the Algorithm 1.

Theorem II.2 (Variant of [3, Theorem 11]). *Let $\delta \ge 3p$ and $4\delta \le d$ and p be a divisor of d and δ. Let $(x_0)_{min} := \min_j |(x_0)_j|$. Then, the estimate x produced in Algorithm 1 with parameters δ and p satisfies*

$$\min_{\theta\in[0,2\pi]} \left\|x_0 - e^{i\theta} x\right\|_2 \le$$

$$C \left[\frac{\|x_0\|_\infty}{(x_0)_{min}^2}\right] \left[\frac{d}{\delta}\right]^2 \sigma_{min}^{-1} \|\mathbf{n}\|_2 + C d^{\frac{1}{4}} \sqrt{\sigma_{min}^{-1} \|\mathbf{n}\|_2},$$

where $\sigma_{min} > 0$ is the minimal singular value of the operator $\mathcal{A}\big|_{T_\delta^p(\mathcal{H}^d)}$, $\mathbf{n}$ is a noise vector and $C > 0$ is an absolute constant.

Note, that Theorem II.2 resembles the conditions and bounds of the its unit length analogue [1]. The only major difference is condition $\delta \geq 3p$, compared to $\delta \geq 3$ in the unit shift length, meaning that the measurement frame should overlap with at least 2 consequent frames.

C. Deterministic measurement masks construction

It remains to construct masks so that $\mathcal{A}\big|_{T_\delta^p(\mathbb{C}^{d\times d})}$ is invertible. There are $d(2\delta - p)$ non-zero entries in a matrix from $T_\delta^p(\mathcal{H}^d)$, what results into condition $K \geq p(2\delta - p)$. We provide two suitable constructions.

1) Standard basis masks:
Let e_i be the i-th standard basis vector in $\mathbb{R}^d$. Consider measurement masks

$$m_{A,k} = e_k,\ m_{C,Re,k,j} = e_k + e_j,$$
$$m_{C,Im,k,j} = e_k - ie_j, \quad k \in [p], k+1 \leq j \leq \delta$$

Theorem II.3. *Consider masks as above. Set $D = d(2\delta - p)$. Let $M' \in \mathbb{C}^{D\times D}$ be the matrix representing measurement mapping $\mathcal{A} : T_\delta^p(\mathcal{H}^d) \mapsto \mathbb{C}^D$. Then, the condition number and the smallest singular value of M' satisfy, respectively,*

$$\kappa(M') \leq c\delta, \text{ and } \sigma_{min}(M') \geq 1/(0.5 + \sqrt{2\delta}),$$

for some constant $c > 0$.

2) Exponential decay masks:
Consider masks

$$(m_j^z)_k = \begin{cases} \frac{1}{\sqrt[4]{2\delta-1}} e^{-k/a_z} e^{\frac{2\pi i (j-1)(k-1)}{2\delta-1}}, & \text{if } 1 \leq k \leq \delta, \\ 0, & \delta < k \leq d, \end{cases}$$

for $z \in [p], j \in [2\delta - 1]$. Such masks are inspired by ptychography. The complex exponential and the fraction correspond to the discrete Fourier transform and e^{-k/a_z} embodies the illumination functions. This construction is similar to the one in [2], and generalizes it by allowing for different intensities a_z. In practice, it can be achieved by changing the light intensity of the beam or by using different lenses for the beam concentration.

Theorem II.4. *Let p be a divisor of δ. Consider masks as above and let the intensity parameters a_z satisfy the conditions*

$$a_z^{-1} - a_{z-1}^{-1} \geq 0.5 \log 2, \quad z = 2, \ldots, p$$
$$a_1^{-1} \geq \frac{p}{p+1}(a_p^{-1} - a_1^{-1}) + \frac{3}{4(p+1)} \log p + \frac{p+2}{p+1} \log 2.$$

Let $M' \in \mathbb{C}^{d(2\delta-1)\times d(2\delta-p)}$ be the matrix representing the measurement mapping $\mathcal{A} : T_\delta(\mathcal{H}^d) \mapsto \mathbb{C}^{d(2\delta-1)}$. Then, M' has rank $d(2\delta - p)$.

Note that we need more than $d(2\delta - p)$ measurements masks. This requirement is related to the technical difficulties in the proof of the theorem. Therefore, in order to prevent the unsolvable system in the case of noise, the Moore-Penrose inverse should be used in the step 1 of Algorithm 1. Also, the requirements for the intensity parameters are different from [2]. While it is sufficient to have $a_1 = \max\{4, (\delta - 1)/2\}$ for the unit length shifts model, condition 2 in the statement implies $a_1 \leq 1/\log 2$ and together with condition 1, it grants us $a_z \leq 1/\log 2$ for all $z \in [p]$.

III. Sketches of Proofs

A. Theorem II.2

Since in [3] only angular synchronization via semi definite optimization is discussed in detail, we include a proof sketch of the eigenvalue based method for completeness. Additionally, in [3], guarantees depend on the term $\tau_G(d, \delta, p)$. The analog of this in our proof can be calculated exactly as p is a divisor of δ. The proof replicates the steps of its unit shift analogue [1]. First, we split the error term into two parts corresponding to the phase and the amplitude errors.

$$\min_{\theta\in[0,2\pi]} \left\| x_0 - e^{i\theta} x \right\|_2 \leq$$
$$\|x_0\|_\infty \cdot \min_{\theta\in[0,2\pi]} \left\| \tilde{x}_0 - e^{i\theta} \tilde{x} \right\|_2 + \| |x_0| \circ \tilde{x} - |x| \circ \tilde{x} \|_2,$$

where $\circ$ denotes the Hadamard product.

The amplitude error can be bounded from above by $C_1 \sqrt[4]{d} \sqrt{\sigma_{min}^{-1} \|\mathbf{n}\|_2}$, for some constant $C_1 > 0$, using Lemma 3 from [2]. For the analysis of the phase error, we need to introduce several definitions. We define $U := T_\delta^p(\mathbb{1}_{d\times d})$ with $\mathbb{1}_{a\times b}$ denoting $a \times b$ matrix with all the entries equal to 1. Next, consider a graph with vertex set $V := [d]$ and adjacency matrix $U - I$. Define the degree matrix $D := \operatorname{diag}(\{\deg(i)\}_{i\in V})$ and the Laplacian $L := I - D^{-1/2} U D^{-1/2}$. Let $0 = \lambda_1(L) \leq \lambda_2(L) \leq \ldots \leq \lambda_d(L)$ be the eigenvalues of L. Then, the spectral gap is defined as $\tau := \lambda_2(L)$. If $\tau > 0$, then by Theorem 3 in [1] the phase error can be bounded as

$$\min_{\theta\in[0,2\pi]} \left\| \tilde{x}_0 - e^{i\theta} \tilde{x} \right\|_2 \leq \frac{C_2 \left\| \tilde{X} - \tilde{X}_0 \right\|_F}{\tau \sqrt{\min_{i\in V} \deg(i)}},$$

where $\tilde{X}_0 := X_0 / |X_0|$ is the entrywise normalization of the non-zero entries of X_0 and $C_2 > 0$ is some constant.

Due to p being a divisor of δ, U has a $p \times p$ block circular structure with blocks either $\mathbb{1}_{d\times d}$ or $\mathbf{O}_{d\times d}$, where $\mathbf{O}_{a\times b}$ is zero $a \times b$ matrix. It follows that $D = (2\delta - p - 1)I$ and $\min_{i\in V} \deg(i) = 2\delta - p - 1$. Further,

$$L = I - D^{-1/2}(U - I)D^{-1/2} = \frac{2\delta - p}{2\delta - p - 1} I - \frac{1}{2\delta - p - 1} U,$$

so that $\tau = (2\delta - p - \lambda_2(U))/(2\delta - p - 1)$. To find the eigenvalues of U, we block-diagonalize U via the block-Fourier transform and obtain blocks $\hat{J}_k = \eta_k \mathbb{1}_{p\times p}$ for $k \in [d']$, where

$$\eta_k := 1 + 2 \sum_{j=1}^{\delta'-1} \cos\left(2\pi j(k-1)/d'\right).$$

Each $\hat{J}_k$ is a rank 1 matrix with non-zero eigenvalue $\mu_k := p\eta_k$ and $\mu_1 = 2\delta - p$. Using Lemma 2 in [1], under conditions of Theorem II.2, for some constant $C_3 > 0$, it holds that

$$\tau \geq \min_{k=2,\ldots,d'} \frac{\mu_1 - |\mu_k|}{2\delta - p - 1} \geq C_3 p \frac{(\delta')^3}{\delta (d')^2} = C_3 \frac{\delta^2}{d^2} > 0.$$

Finally, we bound

$$\left\|\tilde{X}-\tilde{X}_0\right\|_F \leq C_4 \frac{\sigma_{min}^{-1}\|\mathbf{n}\|_2}{\sqrt{\delta d}}\left\|\tilde{X}_0\right\|_F,$$

for some constant $C_4>0$ by an analogue of Lemma 6 from [1] and notice that $\left\|\tilde{X}_0\right\|_F=\sqrt{d(2\delta-p)}$. Combining all established bounds yields the statement of the theorem.

B. Theorem II.3

The goal of the proof is to establish an easy-to-analyze linear system between the entries of the matrix $X \in T_\delta^p(\mathcal{H}^d)$ and the measurements. We will use the notation $i_p := i+1-\lfloor \frac{i}{p} \rfloor p$ and $x^\phi := xx^*,\ x \in \mathbb{C}^d$. Consider the measurement operator $\mathcal{B}: \mathbb{C}^{d\times d} \mapsto \mathbb{C}^D$ defined as

$$\begin{aligned}
(\mathcal{B}(X))_{i,1} &= \langle X, (S_{\lfloor \frac{i}{p}\rfloor p}\ m_{A,i_p})^\phi\ \rangle_F,\\
(\mathcal{B}(X))_{i,2k} &= \langle X, (S_{\lfloor \frac{i}{p}\rfloor p}\ m_{C,Re,i_p,i_p+k})^\phi\ \rangle_F,\\
(\mathcal{B}(X))_{i,2k+1} &= \langle X, (S_{\lfloor \frac{i}{p}\rfloor p}\ m_{C,Im,i_p,i_p+k})^\phi\ \rangle_F,
\end{aligned}$$

$k \in [\delta-1], \exists z \in [d']_0:\ (i+1, i+1+k) \in [zp+1, zp+\delta]^2$,

$$\begin{aligned}
(\mathcal{B}(X))_{i,2k} &= (\mathcal{B}(X))_{i,1} + (\mathcal{B}(X))_{i+k,1},\\
(\mathcal{B}(X))_{i,2k+1} &= (\mathcal{B}(X))_{i,1} + (\mathcal{B}(X))_{i+k,1},
\end{aligned}$$

$k \in [\delta-1], \nexists z \in [d']_0:\ (i+1, i+1+k) \in [zp+1, zp+\delta]^2$

for all $i \in [d]_0$. For $X \in T_\delta^p(\mathcal{H}^d)$ it holds that for all $j \in [d]_0, k \in [\delta-1]$

$$\begin{aligned}
X_{j+1,j+1} &= \mathcal{B}(X)_{j,1},\\
X_{j+1,j+1+k} &= \frac{1}{2}\mathcal{B}(X)_{j,2k} + \frac{i}{2}\mathcal{B}(X)_{j,2k+1}\\
&\quad - \frac{1+i}{2}\left(\mathcal{B}(X)_{j,1} + \mathcal{B}(X)_{j+k,1}\right).
\end{aligned}$$

The established system can be analyzed as in [1, pp.8-9], which yields Theorem II.3.

C. Theorem II.4

The first steps of the proof replicate its unit shift analogue (up to S_k matrices), but later technical complications arise from the fact that different windows represented by different intensities a_z are combined. More precisely, the structure of the measurement matrix is given as follows. Define $(2\delta-1)\times(2\delta-2q+1)$ matrices $M_{s,q}^z, s \in [\delta'], q \in [p], z \in [p]$ with entries in a row $j \in [2\delta-1]$ given by

$$\begin{aligned}
(M_{s,q}^z)_{j,i} &= (m_j^z)_{\delta-i+1+(s-1)p}\overline{(m_j^z)}_{(s-1)p+q},\\
(M_{s,q}^z)_{j,2\delta-2q+2-i} &= (m_j^z)_{(s-1)p+q}\overline{(m_j^z)}_{\delta-i+1+(s-1)p},
\end{aligned}$$

when $(s-1)p+1 \leq i \leq \delta-q+1$ and 0 otherwise.

Next, assemble the obtained matrices into $p(2\delta-1)\times p(2\delta-p)$ blocks $M_s, s \in [\delta']$ as

$$M_s = \begin{bmatrix} M_{s,1}^1 & \dots & M_{s,p}^1 \\ \vdots & \ddots & \vdots \\ M_{s,1}^p & \dots & M_{s,p}^p \end{bmatrix}, \qquad \text{(III.1)}$$

and, finally, the $d(2\delta-1)\times d(2\delta-p)$ full measurement matrix as

$$M = \begin{bmatrix} M_1 & M_2 & \dots & M_{\delta'} & 0 & 0 & \dots & 0 \\ 0 & M_1 & \dots & M_{\delta'-1} & M_{\delta'} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ M_2 & M_3 & \dots & 0 & 0 & 0 & \dots & M_1 \end{bmatrix}.$$

We can vectorize $X \in T_\delta^p(\mathcal{H}^d)$, so that measurements y take the form $M \cdot \mathrm{vec}(X) = y$. M can be block-diagonalized with blocks J_k, each itself consist of the block matrices $J_{k,q}^z, k \in [d'], q \in [p], z \in [p]$, which we decompose as follows.

$$J_{k,q}^z = F \cdot \begin{bmatrix} \mathbf{O}_{(q-1)\times(2\delta-2q+1)} \\ S_{k,q}^z \\ \mathbf{O}_{(q-1)\times(2\delta-2q+1)} \end{bmatrix} =: F \cdot \tilde{S}_{k,q}^z,$$

where $S_{k,q}^z$ is a $(2\delta-2q+1)\times(2\delta-2q+1)$ diagonal matrix and F is the $2\delta-1$ discrete Fourier transform matrix.

The positioning of the blocks $J_{k,q}^z$ inside J_k is the same as the positioning of the $M_{s,q}^z$ in (III.1). That is, with $\tilde{F} := \mathrm{diag}(F,\dots,F)$, we obtain

$$J_k = \begin{bmatrix} J_{k,1}^1 & \dots & J_{k,p}^1 \\ \vdots & \ddots & \vdots \\ J_{k,1}^p & \dots & J_{k,p}^p \end{bmatrix} = \tilde{F}\cdot \begin{bmatrix} \tilde{S}_{k,1}^1 & \dots & \tilde{S}_{k,p}^1 \\ \vdots & \ddots & \vdots \\ \tilde{S}_{k,1}^p & \dots & \tilde{S}_{k,p}^p \end{bmatrix} =: \tilde{F}\cdot\tilde{S}_k$$

Permuting the rows and columns of $\tilde{S}_k$ leads to a block-diagonal matrix H_k whose subblocks $H_{k,q}, q \in [2\delta-1]$ satisfy the symmetry relation $H_{k,q} = H_{k,2\delta-q}$. In order to state the structure of $H_{k,q}$ we need to introduce supplementary matrices. Due to space constraints, we only state some important properties of these matrices. Let $D_{k,q}$ and $R_{k,q}$ be invertible diagonal matrices, $T_{k,q}$ invertible matrix and V_a be the $a\times p$ Vandermonde matrix with distinct generating entries. For $q \in [p]$, $H_{k,q} = D_{k,q}V_q$ is a rank q matrix. For $p+1 \leq q \leq 2p$, $H_{k,q} = D_{k,q}V_pT_{k,q}$ is a rank p matrix. Finally, for $2p+1 \leq q \leq \delta$, $q = zp+w+1$, $z \in [\delta'-1]$, $w \in [p]_0$, it holds that $H_{k,q} = D_{k,q}\left(V_p - R_{k,q}\cdot\begin{bmatrix}\mathbf{O}_{p\times(w+1)} & V_{p-w-1}\end{bmatrix}\right)$. The application of inequalities for singular values united with conditions on the intensities a_z justifies that $H_{k,q}$ has rank p. Thus, H_k has rank $p(2\delta-p)$. As the permutation, $\tilde{F}$ and the block Fourier are unitary transformations, the rank of M remains $d(2\delta-p)$, which concludes the proof.

IV. Conclusions and further steps

To summarize, we derived recovery guarantees for a generalized BPR ptychography algorithm with local correlation measurements of fixed shift length for new classes of masks. The next steps will be numerical tests of the algorithm and its comparison to other solutions. Also, an important goal will be to generalize the approach used for the proof of the exponential decay masks to the mask actually used in ptychographical measurements (I.1).

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