Optimization in the construction of nearly cardinal and nearly symmetric wavelets

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***Abstract*—We present two approaches to the construction of scaling functions and wavelets that generate nearly cardinal and nearly symmetric wavelets on the line. The first approach casts wavelet construction as an optimization problem by imposing constraints on the integer samples of the scaling function and its associated wavelet and with an objective function that minimizes deviation from cardinality or symmetry. The second method is an extension of the feasibility approach by Franklin, Hogan, and Tam to allow for symmetry by considering variables generated from uniform samples of the quadrature mirror filter, and is solved via the Douglas-Rachford algorithm.**

I. Wavelets on the line, Cardinality, and Symmetry

Shortly after Mallat and Meyer [1], [2] introduced the concept of multiresolution analysis, Daubechies [3] derived a special class of real and compactly supported orthonormal dyadic wavelets with reasonable smoothness. While smooth wavelets with compact support and orthonormal shifts have been useful in many applications, cardinality or symmetry is also often sought. However, cardinality cannot be achieved when coupled with compact support, continuity, and orthogonal shifts [4]. Similarly, other than the case of the Haar wavelet, symmetry is unattainable while maintaining real-valuedness, orthogonality, and compact support [3]. In this section, we introduce wavelets on the line and motivate our search for nearly cardinal and nearly symmetric scaling functions and wavelets.

A. Scaling functions and wavelets

For $f \in L^1(\mathbb{R})$, we define the Fourier transform of f by $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}\, dx$, and for $f \in L^2(\mathbb{R})$ by a limiting process. The standard method of constructing orthonormal wavelet bases is through multiresolution analysis which is described below.

Definition 1: A *multiresolution analysis* (MRA) for $L^2(\mathbb{R})$ consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ and a function $\varphi \in V_0$ such that the following conditions hold:

(i) the spaces V_j are nested, i.e., $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,
(ii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,
(iii) $f(x) \in V_0$ if and only if $f(x-k) \in V_0$ for all $k \in \mathbb{Z}$,
(iv) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$ for all $j \in \mathbb{Z}$, and
(v) $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ forms an orthonormal basis for V_0.

Under these circumstances, there exists $\{h_k\} \in \ell_2(\mathbb{Z})$ so that φ satisfies a *scaling equation* of the form

$$\frac{1}{2}\varphi\left(\frac{x}{2}\right) = \sum_{k\in\mathbb{Z}} h_k \varphi(x-k). \tag{1}$$

We call φ a *scaling function*. Taking the Fourier transform of both sides of (1) gives $\hat{\varphi}(2\xi) = H(\xi)\hat{\varphi}(\xi)$, where $H(\xi) = \sum_k h_k e^{-2\pi i k\xi}$ is called the *scaling filter*. It satisfies what is called the *quadrature mirror filter property* given by

$$\left|H(\xi)\right|^2 + \left|H\left(\xi+\frac{1}{2}\right)\right|^2 \equiv 1, \tag{2}$$

and so H is also called a *quadrature mirror filter (QMF)*. Moreover, we can find a *wavelet function* ψ satisfying

$$\frac{1}{2}\psi\left(\frac{x}{2}\right) = \sum_{k\in\mathbb{Z}} g_k \varphi(x-k), \tag{3}$$

where $\{g_k\}_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$. In the Fourier domain, (3) becomes $\hat{\psi}(2\xi) = G(\xi)\hat{\varphi}(\xi)$, where $G(\xi) = \sum_k g_k e^{-2\pi i k\xi}$. Note that H and G are together called a QMF pair. They satisfy

$$H(\xi)\overline{G(\xi)} + H\left(\xi+\frac{1}{2}\right)\overline{G\left(\xi+\frac{1}{2}\right)} \equiv 0. \tag{4}$$

If H is a QMF, then $G(\xi) = -e^{-2\pi i\xi}\overline{H(\xi+1/2)}$ is a possible choice that makes H and G a QMF pair.

For the scaling function φ and wavelet ψ to have P continuous and bounded derivatives, the following necessary conditions are usually imposed:

$$\left.\frac{d^k H(\xi)}{d\xi^k}\right|_{\xi=\frac{1}{2}} = 0 \tag{5}$$

for all $k \in \{0, 1, 2, \ldots, P\}$.

B. Cardinality

A scaling function φ is said to be *cardinal* at $P \in \mathbb{Z}$ if $\varphi(k) = \delta_{kP}$, for all $k \in \mathbb{Z}$. A famous example is the cardinal sine function φ_S which is a scaling function for the Shannon MRA. The Shannon sampling theorem describes the recovery of functions in the closed linear span of integer shifts of φ_S from their integer samples. This property of admitting a reconstruction formula based on integer samples is typical for cardinal scaling functions. Walter [5] extended the idea to other wavelet spaces by giving a reconstruction formula that represents a function $f \in V_0$ in terms of its samples at the integers and integer translates of an interpolant derived from a scaling function that may not be cardinal, and was furthered by Janssen [6] to non-integer samples.

However, since cardinality of a scaling function is inconsistent with compact support, orthogonal shifts, and continuity, we instead search for nearly cardinal scaling functions. For scaling functions that are nearly cardinal, an invertible sampling operator may be defined for functions in V_0.

Definition 2: A scaling function $\varphi \in V_0$ is *nearly cardinal* at $P \in \mathbb{Z}$ when $\gamma = \gamma(\varphi) := \sum_k |\varphi(k) - \delta_{kP}| < 1$.

We refer to γ as the *measure of cardinality*. Note that φ is cardinal whenever $\gamma = 0$. The following result appears in [7].

Theorem 1: If $f \in V_0$, the sampling operator S_P defined as

$$S_P f(x) = \sum_k f(k)\varphi(x - k - P) \tag{6}$$

is invertible whenever φ is nearly cardinal.

Furthermore, it is shown in [7] that if φ is nearly cardinal, then $f \in V_0$ can be recovered from its integer samples $\{f(k)\}_{k=-\infty}^{\infty}$. In fact, the sequence $\{f_n\}_{n=0}^{\infty} \subset V_0$ given by $f_0 = S_P f$, $f_{n+1} = f_n + S_P(f - f_n)$, converges to f.

C. Symmetry

A scaling function φ is *symmetric* about $P \in \mathbb{R}$ if $\varphi(x) = \varphi(2P - x)$ for any $x \in \mathbb{R}$. However, symmetry cannot be coupled with real-valuedness, compact support, orthogonal shifts, and continuity. If the real-valuedness condition is lifted, or when the orthogonality property is relaxed, then symmetry is attainable. Additionally, the length of support and regularity can be traded off for symmetry, as in the case of the coiflets [3]. While asymmetric wavelets work well in a number of numerical applications, quantization errors in image coding are prominent around the edges of images. With our visual system more tolerant of symmetric errors than asymmetric ones, less asymmetry could allow for greater compressibility as some perceptual errors are neglected [3].

To minimize asymmetry, we first look at the following characterizations of symmetric scaling functions [7].

Proposition 2: The following are equivalent:

(i) $\varphi(x) = \varphi(2P - x)$
(ii) $H(\xi) = e^{-4\pi i P\xi} H(-\xi)$
(iii) $h_k = h_{2P-k}$

In the next section, we present an optimization problem whose objective function is derived from one of the equivalent statements above. Minimizing this objective function will allow us to obtain nearly symmetric scaling functions.

II. QMF construction using the Zak transform

A. QMF construction based on integer samples

The Zak transform is a signal transform yielding a time-frequency representation of time-continuous signals sampled at a uniform rate. For a more general discussion on the Zak transform and its properties, see [3], [8], [9]. Given a function f in the Schwarz space, we define the *Zak transform* Zf of f by

$$Zf(x,\xi) = \sum_{k\in\mathbb{Z}} f(x+k)e^{2\pi ik\xi} \tag{7}$$

where $(x, \xi) \in \mathbb{R}^2$. We can further define the *complexified Zak transform* $Z_{\mathbb{C}}f(x, z)$ by replacing the exponential $e^{2\pi i\xi}$ in (7) with a complex number z. The complexified Zak transform is given by the Laurent series

$$Z_{\mathbb{C}}f(x,z) = \sum_k f(x+k)z^k, \tag{8}$$

whenever $x \in \mathbb{R}$ and $z \in \mathbb{C}$. The definitions of Z and $Z_{\mathbb{C}}$ may be extended to $L^2(\mathbb{R})$. Henceforth, we abuse notation by omitting the subscript $\mathbb{C}$, and switching between (7) and (8) at our convenience.

We now derive a QMF construction that imposes certain conditions on the integer samples of φ and ψ using the Zak transform.

If F is a Laurent polynomial of the form $F(z) = \sum_k a_k z^k$, we define $F^*(z) = \bar{F}(z^*) = \sum_k \bar{a_k} z^{-k}$, where $z^* = \frac{1}{\bar{z}}$. Given a Laurent polynomial F, we define an operator T_F on Laurent polynomials f by $T_F f(z^2) = F(z)f(z) + F(-z)f(-z)$. Let $\Phi(z) = Z_{\mathbb{C}}\varphi(0, z)$ and $\Psi(z) = Z_{\mathbb{C}}\psi(0, z)$, respectively.

In the Zak domain, equations (1) and (3) may be written as

$$Z_{\mathbb{C}}\varphi(x,z^2) = H(z)Z_{\mathbb{C}}\varphi(2x,z) + H(-z)Z_{\mathbb{C}}\varphi(2x,-z) \tag{9}$$

$$Z_{\mathbb{C}}\psi(x,z^2) = G(z)Z_{\mathbb{C}}\varphi(2x,z) + G(-z)Z_{\mathbb{C}}\varphi(2x,-z). \tag{10}$$

For the details of the proof of equations (9) and (10), the reader is referred to [8]. Note that given a QMF H, a possible choice for the conjugate QMF G is given by $G(z) = -z^{2Q+1}H^*(-z)$, where the role of Q is to re-center the wavelet ψ. In particular, if φ is compactly supported on $[0, M]$, we can choose $Q = \frac{M-1}{2}$ so that ψ is also supported on the same interval. Setting $x = 0$ in (9) and (10), we obtain

$$H(z) = \frac{\Phi(z^2)\Phi^*(z) + z^{2Q+1}\Phi(-z)\Psi^*(z^2)}{\Phi(z)\Phi^*(z) + \Phi(-z)\Phi^*(-z)}. \tag{11}$$

From (11) we see that H is written in terms of the integer samples of φ and ψ. We define the numerator of the fraction on the right hand side of (11) as $B(z)$, and the denominator as $C(z)$. C then satisfies $C(z) = C(-z) = C^*(z) = C^*(-z)$ and $C(z) = \Phi(z^2)\Phi^*(z^2) + \Psi(z^2)\Psi^*(z^2)$. Moreover, the expression we have derived for $H(z)$ in (11) satisfies the QMF condition in (2). Additionally, we need extra constraints to account for compact support and regularity conditions. In the next proposition, we include these conditions while summarizing the construction discussed above.

Theorem 3: For odd $M > 1$, let $\Phi(z) = \sum_{k=1}^{M-1} \varphi(k)z^k$, $\Psi(z) = \sum_{k=1}^{M-1} \psi(k)z^k$. Let $B(z)$, $C(z)$ be defined as above, and set

$$A(z) := \Phi(z)\Phi^*(z)+\Phi(-z)\Phi^*(-z)-\Phi(z^2)\Phi^*(z^2)-\Psi(z^2)\Psi^*(z^2).$$

Suppose further that the following conditions hold:

1) $\Phi(1) = 1$,
2) $\Psi(1) = \Phi(-1)$,
3) $A(z) \equiv 0$, and
4) $B'(-1) = B''(-1) = \cdots = B^{(\frac{M-1}{2})}(-1) = 0$.

Then $H(z) = \dfrac{B(z)}{C(z)}$ satisfies:

(a) $H'(-1) = H''(-1) = \cdots = H^{(\frac{M-1}{2})}(-1) = 0$, and
(b) $T_H\Phi = \Phi$ and $T_G\Phi = \Psi$

where $G(z) = -z^{2Q+1}H^*(-z)$.

The conditions of this theorem identify the exact set of constraints that must be imposed in constructing a QMF.

B. Near cardinality as an optimization problem

Combining the results in Theorem 3 and the definition of near cardinality lead us to the following optimization problem.

Problem 1: Minimize $\gamma(\varphi) = \sum_{k=1}^{M-1} |\varphi(k) - \delta_{kP}|$ subject to conditions (1) to (4) of Theorem 3.

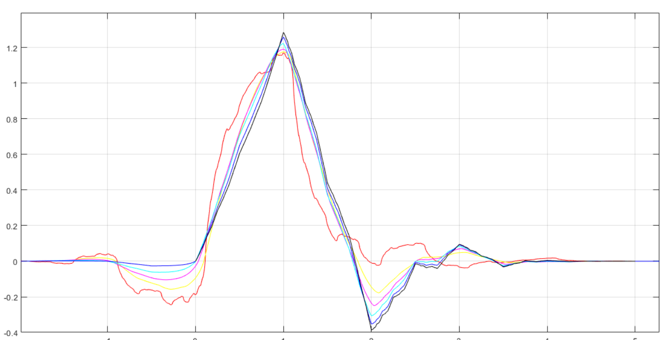


Fig. 1. Examples of nearly cardinal scaling functions generated by solving Problem 1 with $P = 1$ and $M = 5$. The plots in black, blue, cyan, magenta, yellow, and red (in that order) have decreasing measures of cardinality. The graph in black also corresponds to Daubechies' ${}_3\phi$. For more details about the construction of these examples, see [7].

C. Near symmetry as an optimization problem

For symmetry, we first investigate if $\varphi(2P-k) = \varphi(k)$ for a $P \in \mathbb{R}$, $\forall\, k \in \mathbb{Z}$ will imply that φ is symmetric. We claim that H given in (11) will satisfy $H(z) = z^{2P}H\left(\frac{1}{z}\right)$ whenever $\varphi(2P-k) = \varphi(k)$. The following lemmata and theorem from [7] show that the claim is true.

Lemma 4: If $\varphi(2P - x) = \varphi(x)$, then $\psi(x) = (-1)^{2P}\psi(2Q + 1 - x)$.

Lemma 5: If $\varphi(2P-k) = \varphi(k)$ and $\psi(k) = (-1)^{2P}\psi(2Q+1-k)$ for $k \in \mathbb{Z}$, then the following statements are true:

1) $\Phi(z) = z^{2P}\Phi(\frac{1}{z})$
2) $\Psi(z) = (-1)^{2P}z^{2Q+1}\Psi(\frac{1}{z})$
3) $C(z) = C(\frac{1}{z})$
4) $B(z) = z^{2P}B(\frac{1}{z})$

Theorem 6: Suppose $\varphi(2P - k) = \varphi(k)$, then $H(z) = z^{2P}H\left(\frac{1}{z}\right)$, i.e., φ is symmetric.

Theorem 6 implies that if the integer samples of φ are even about P, then the scaling filter H in Theorem 3 satisfies the second statement of Proposition 2, thus establishing the symmetry of φ. At this point, the problem of finding nearly symmetric scaling functions may now be viewed as an optimization problem. In particular, if we seek symmetry at the center of support $[0, M]$, we have the following problem.

Problem 2: Minimize $\sum_{k=1}^{(M-1)/2} |\varphi(k) - \varphi(M-k)|$ subject to conditions (1) to (4) of Theorem 3.

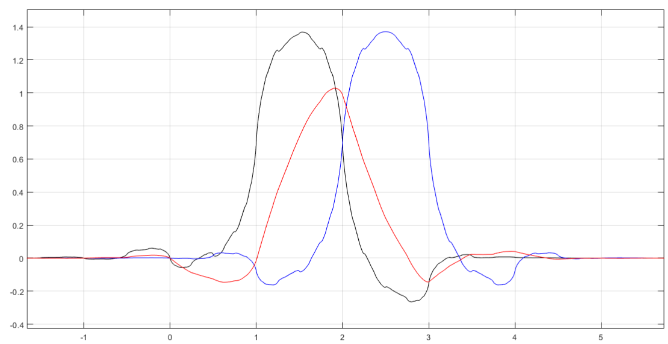


Fig. 2. Examples of nearly symmetric scaling functions about different points: the graph in black about $P = 1.5$, in blue about $P = 2.5$ (Problem 2), and in red about $P = 2$; with $M = 5$ in all cases. For more details about the construction of these examples, see [7].

We note that a scaling function produced as a solution of Problem 1 or 2 may not be compactly supported. That Φ and Ψ are Laurent polynomials is only a necessary condition for H to be trigonometric polynomial – there is no reason to believe that C will always divide B in (11). Hence, H may be a rational function rather than a polynomial. But in such cases, the scaling filter may be truncated to make it a good starting point for a Douglas-Rachford iteration to solve the feasibility problem given in the next section.

III. QMF construction using feasibility approach

For the second approach, we refer to [10] and [11] where the problem of constructing a one-dimensional scaling function φ and wavelet ψ of support length $M - 1$ (where M is even) is recast as a feasibility problem over three constraint sets. For consistency in notation, we re-define our scaling and wavelet filters by $H(\xi) := \sum_{k=0}^{M-1} h_k e^{2\pi i k\xi}$ and $G(\xi) = \sum_{k=0}^{M-1} g_k e^{2\pi i k\xi}$, respectively. Given that

$$U(\xi) = \begin{bmatrix} H(\xi) & G(\xi) \\ H\left(\xi + \frac{1}{2}\right) & G\left(\xi + \frac{1}{2}\right) \end{bmatrix},$$

the feasibility approach finds a matrix-valued trigonometric polynomial of the form $U(\xi) = \sum_{k=0}^{M-1} A_k e^{2\pi i k\xi}$ that satisfies the following:

1) $U(\xi)$ is unitary for almost every ξ,
2) $U(0) = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}$ where $|z| = 1$, and
3) $\frac{d^l}{d\xi^l} U(\xi)\big|_{\xi=0}$ is diagonal, $0 \le l \le \frac{M-2}{2}$.

We can use Proposition 2 to write a symmetry constraint in terms of $U(\xi)$. For brevity, if $A \in \mathbb{C}^{2\times 2}$, we define $A^\dagger$ as taking copy of A with negated off-diagonal entries. The symmetry condition is as follows.

Theorem 7: If the scaling function φ is symmetric about the center of support, then $U(\xi) = e^{2\pi i (M-1)\xi} U(\xi)^\dagger$.

When $M = 6$, the feasibility approach (with a clever choice of starting point) was able to generate a symmetric scaling function even without incorporating any condition akin to Theorem 7. However, the generated symmetric example is complex-valued. To force real-valuedness, we add another constraint as given in the next theorem.

Theorem 8: The scaling function φ and the wavelet ψ are real-valued if and only if $U(\xi) = \overline{U(-\xi)}$.

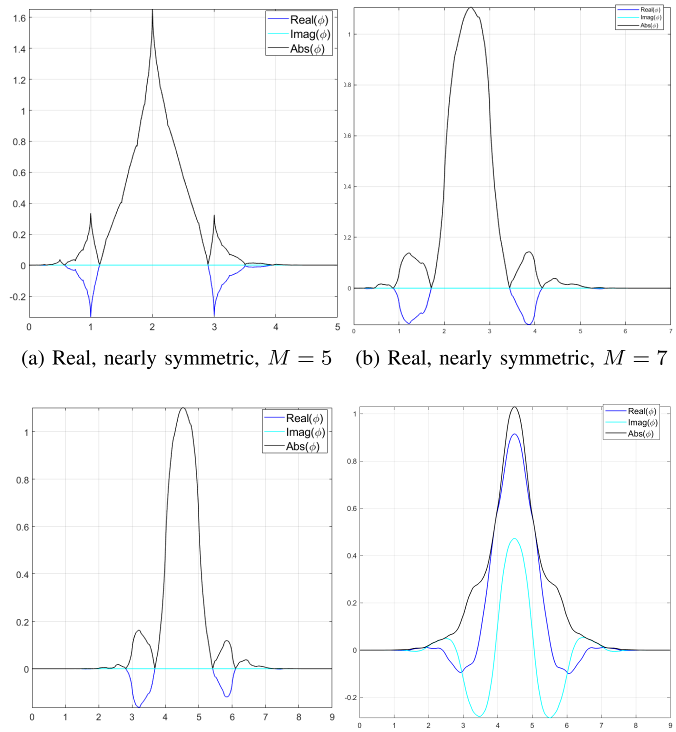
In [10], [11], discretization of the problem is obtained by a uniform sampling performed on $U(\xi)$: let $U_j := U(j/M) = \sum_{k=0}^{M-1} A_k e^{2\pi i jk/M}$. We say $\mathcal{U} := (U_0, U_1, \ldots, U_{M-1}) \in (\mathbb{C}^{2\times 2})^M$ is a *matrix ensemble*. An M-point discrete Fourier transform $\mathcal{F}_M$ provides a (multiple of a) unitary mapping between samples U_j and coefficients A_k, i.e., $A_k = \frac{1}{M}\sum_{k=0}^{M-1} U_j e^{-2\pi i jk/M}$.
Finally, let $(\mathbb{C}^{2\times 2})^M_\sigma$ denote the set of all ensembles $\mathcal{U}$ satisfying the σ condition $U_{j+M/2} = \sigma U_j$ $(0 \le j \le M/2-1)$ where σ is the row swap matrix $\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $(\mathbb{U}^{2\times 2})^M$ is the set of ensembles with unitary entries. Then the feasibility problem with additional symmetry and real conditions is restated as:

Problem 3: For an even integer $M \ge 4$, find $\mathcal{U} \in C := \bigcap_{i=1}^{5} C_i$, where

$C_1 := \{\mathcal{U} \in (\mathbb{C}^{2\times 2})^M_\sigma \cap (\mathbb{U}^{2\times 2})^M : U(0) = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}, |z| = 1\}$,
$C_2 := \left\{\mathcal{U} \in (\mathbb{C}^{2\times 2})^M_\sigma : (\mathcal{F}_{\mathcal{M}} e^{\pi i j/M} (\mathcal{F}_M^{-1})(U))_j, 0 \le j \le \frac{M-2}{2}\right\}$,
$C_3 := \{\mathcal{U} \in (\mathbb{C}^{2\times 2})^M_\sigma : \sum_{j=0}^{M-1} j^l (\mathcal{F}_M U)_j \in \mathrm{diag}(\mathbb{C})^{2\times 2}\}$,
$C_4 := \{\mathcal{U} \in (\mathbb{C}^{2\times 2})^M_\sigma : ||U_j - e^{2\pi i (M-1)j/M} U_j^\dagger|| \le \epsilon\}$, and
$C_5 := \{\mathcal{U} \in (\mathbb{C}^{2\times 2})^M_\sigma : U_j = \overline{U_{M-j}}\}$.

Constraint C_4 depends on a tolerance ϵ which measures the symmetry of the result. We may set $\epsilon = 0$ and turn off C_5 when seeking a complex-valued symmetric scaling function. Moreover, Theorem 7 may also be reformulated to allow for symmetry about a point other than the center of support.

It is also possible to write down a condition for cardinality in terms of the wavelet matrix $U(\xi)$. However, at the moment, compactly supported nearly cardinal scaling functions have been successfully obtained by feeding to the Douglas-Rachford algorithm the truncated versions of nearly cardinal scaling functions obtained using the first approach.



(a) Real, nearly symmetric, $M = 5$ (b) Real, nearly symmetric, $M = 7$

(c) Real, nearly symmetric, $M = 9$ (d) Complex, symmetric, $M = 9$

Fig. 3. Examples of nearly symmetric real-valued scaling functions, and a symmetry complex-valued function obtained using the feasibility approach.

IV. Acknowledgements

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