A quantitative Balian-Low theorem for subspaces

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***Abstract*—We consider Gabor Riesz sequences generated by a window function with finite uncertainty product over a rational lattice in $\mathbb{R}^2$. We prove that the distance of a time-frequency shift of the window function to the Gabor space is equivalent, up to constants, to the Euclidean distance of the parameters of the time-frequency shift to the lattice. Under certain additional assumptions, these constants can be estimated. As a byproduct of the methods employed, we also obtain a strengthening of the so-called weak Balian-Low theorem.**

I. Introduction

This paper as well as the sister paper written by the same authors, [4], are research extensions of the article [3]. The present paper discusses extensions in the classical Balian-Low setting of Gabor windows with finite uncertainty product, whereas [4] deals with the amalgam Balian-Low setting of Gabor windows in the Feichtinger algebra $\mathcal{S}_0$.

The classical Balian-Low theorem provides one of the fundamental restrictions of time-frequency analysis. A Gabor system $(g, \Lambda) = \{e^{2\pi i b x} g(x-a) : (a,b) \in \Lambda\}$ with the window $g \in L^2(\mathbb{R})$ having finite uncertainty product, i.e., $\int_{\mathbb{R}} x^2 |g(x)|^2 dx \cdot \int_{\mathbb{R}} \xi^2 |\hat{g}(\xi)|^2 < \infty$, and $\Lambda \subset \mathbb{R}^2$ a lattice can never be a Riesz basis for $L^2(\mathbb{R})$.

A general result [8, Cor. 7.5.2] states that a Gabor system that is a Riesz basis for $L^2(\mathbb{R})$ must have an underlying lattice with critical density. In this sense, the Balian-Low theorem's restriction on the regularity of the window function only comes into play in the case of Λ having density 1. However, Gabor systems that form Riesz sequences within $L^2(\mathbb{R})$ are still possible if the corresponding lattice densities are strictly smaller than 1. A characterization of Balian-Low type behaviour in such regimes is given by the main result of [3].

Theorem I.1. *Let $g \in L^2(\mathbb{R})$ and let $\Lambda \subset \mathbb{R}^2$ be a rational density lattice such that the Gabor system (g, Λ) as defined above is a Riesz basis for its closed linear span $\mathcal{G}(g, \Lambda)$. If $e^{2\pi i \eta x} g(x-u) \in \mathcal{G}(g, \Lambda)$ for some $(u, \eta) \notin \Lambda$, then g has infinite uncertainty product.*

It is worth mentioning that once the restriction on the density of Λ being equal to 1 is dropped, i.e. if the Gabor space is allowed to be a proper subspace of $L^2(\mathbb{R})$, the density of the lattice can be either rational or irrational. The rational density assumption is a byproduct of the methods used in [5]. An exploration into alternative approaches with partial results is given in [4], [6].

The main focus of this paper is a quantitative version of Theorem I.1 suggested by engineering applications. The principal result reads as follows.

Theorem I.2. *[5, Thm 1.3] Let $\Lambda \subset \mathbb{R}^2$ be a rational density lattice and let $g \in L^2(\mathbb{R})$ have finite uncertainty product such that the Gabor system $\{e^{2\pi i b x} g(x-a) : (a,b) \in \Lambda\}$ is a Riesz basis of its closed linear span $\mathcal{G}(g, \Lambda)$. Then there exist constants $\alpha, \beta > 0$ such that for all $(u, \eta) \in \mathbb{R}^2$ we have*

$$\begin{aligned}\alpha \cdot \operatorname{dist}((u,\eta), \Lambda) &\leq \operatorname{dist}(e^{2\pi i \eta x} g(x-u), \mathcal{G}(g, \Lambda)) \\ &\leq \beta \cdot \operatorname{dist}((u,\eta), \Lambda).\end{aligned}$$

The remainder of this paper is organized as follows. Section II contains preliminaries, notation and general lemmas. Section III contains a sketch of the main result based on the derivative of the time-frequency map and Theorem I.1. Finally, Section IV provides explicit values for the constants α, β under stronger assumptions and discusses a strengthening of the so called weak Balian-Low theorem [9, Thm 8].

II. Preliminaries

We shall denote $\mathbb{H}^1(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : f, \hat{f} \in H^1(\mathbb{R})\}$, where $H^1(\mathbb{R})$ is the usual Sobolev space. The condition of g having finite uncertainty product is equivalent to $g \in \mathbb{H}^1(\mathbb{R})$.

For $f \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$ we define the translation and respectively modulation operators by $T_a f(x) := f(x-a)$ and $M_b f(x) := e^{2\pi i b x} f(x)$. Both operators are unitary on $L^2(\mathbb{R})$ and so is their composition, the time-frequency shift operator with parameters (a,b), defined as $\pi(a,b) := M_b T_a$.

For an invertible 2×2 matrix A we have a non-degenerate lattice $\Lambda = A\mathbb{Z}^2$. The density of Λ is then given by $|\det(A)|^{-1}$. For a function $g \in L^2(\mathbb{R})$, called a window function, we define the Gabor system with respect to the lattice Λ by $(g, \Lambda) := \{\pi(a,b) g : (a,b) \in \Lambda\}$. The $L^2(\mathbb{R})$ closure of the span of the Gabor system is called the associated Gabor space $\mathcal{G}(g, \Lambda) := \overline{\operatorname{span}}(g, \Lambda)$. For a closed linear subspace $\mathcal{G} \subset L^2(\mathbb{R})$ we set $\mathfrak{I}(\mathcal{G}) := \{z \in \mathbb{R}^2 : \pi(z)\mathcal{G} \subset \mathcal{G}\}$ to be the set of time-frequency shifts which leave $\mathcal{G}$ invariant.

Lemma II.1. *[2, Prop. A.1] Let $g \in L^2(\mathbb{R})$ and let $\Lambda \subset \mathbb{R}^2$ be a lattice. Set $\mathcal{G} := \mathcal{G}(g, \Lambda)$. Then $z \in \mathfrak{I}(\mathcal{G})$ if and only if $\pi(z) g \in \mathcal{G}$. Additionally, $\mathfrak{I}(\mathcal{G})$ is a closed additive subgroup of $\mathbb{R}^2$.*

Proposition II.2. *[5, Cor. 3.3] Let $g \in \mathbb{H}^1(\mathbb{R})$ and define the time-frequency map $S_g : \mathbb{R}^2 \to L^2(\mathbb{R})$ by*

$S_g(a,b) := \pi(a,b)g = e^{2\pi ib\cdot}g(\cdot - a)$. *Then the map* S_g *is continuously (Fréchet) differentiable with*

$$S_g'(z)(a,b) = -a\pi(z)g' + 2\pi ibX\pi(z)g, \quad z \in \mathbb{R}^2.$$

Sketch of proof. We denote by X the position operator defined by $Xf(x) := xf(x)$. Using the fact that time-frequency shift operators commute up to unimodular constants, it suffices to show $S_g'(a,b) = -ag' + 2\pi ibXg$, where we use the shorthand notation S_g' for the derivative of the time-frequency map at the origin of the time-frequency plane.

By direct computation we have

$$\begin{aligned} |S_g(a,b) - g + ag' - 2\pi ibXg| = e^{2\pi ibx}(T_a g - g + ag')(x) + \\ + (e^{2\pi ibx} - 1 - 2\pi ibx)g(x) + \\ + a(1 - e^{2\pi ibx})g'(x). \end{aligned} \tag{1}$$

The claim is proved by showing that the L^2 norm of each of the terms divided by $\sqrt{a^2+b^2}$ tends to 0 as $(a,b) \to (0,0)$. For example, for the middle term in (1) convergence is shown by splitting the integral into the norm over the interval centered at the origin of radius $\frac{1}{\sqrt{b}}$ and the rest of the real line and using the inequality $|\operatorname{sinc}(x) - e^{-i\pi x}| \le \min(2, \pi|x|)$ as well as the fact that $Xg \in L^2(\mathbb{R})$. The other two terms in (1) are dealt with similarly. □

Finally we discuss the main technical tool used for the proof of Theorem I.2, the Zak transform [8, Chap. 8]. The dependency of the methodology of the proof on it is the principal reason for assuming the additional restriction of the density of Λ being rational. For a function $f \in L^2(\mathbb{R})$, the Zak transform is defined by the $L^2_{loc}(\mathbb{R}^2)$ limit

$$Zf(x,\omega) := \lim_{N\to\infty} \sum_{k=-N}^{N} e^{2\pi ik\omega} f(x-k), \quad (x,\omega) \in \mathbb{R}^2.$$

The Zak transform $f \mapsto Zf$ is unitary from $L^2(\mathbb{R})$ to $L^2([0,1]^2)$ and exhibits a variety of useful properties. For example, it is quasi-periodic, $Zf(x+m, \omega+n) = e^{2\pi im\omega}Zf(x,\omega)$ for all $m,n \in \mathbb{Z}$ and it maps time-frequency shifts to twisted shifts in the Zak domain, $Z(\pi(u,\eta)f)(x,\omega) = e^{2\pi i\eta x}Zf(x-u, \omega-\eta)$ for all $(u,\eta) \in \mathbb{R}^2$. The Zak transform also satisfies the inversion formulae $f(x) = \int_0^1 Zf(x,\omega)d\omega$ and $\hat{f}(\omega) = \int_0^1 e^{-2\pi ix\omega}Zf(x,\omega)dx$. All of these properties hold for a.e. $(x,\omega) \in \mathbb{R}^2$.

Lemma II.3. *[5, Lemma 2.4] Let* $f \in L^2(\mathbb{R})$. *Then* $f \in \mathbb{H}^1(\mathbb{R})$ *if and only if* $Zf \in H^1_{loc}(\mathbb{R}^2)$. *In this case, the weak partial derivatives of* Zf *are given by* $\partial_1 Zf = Z(f')$ *and* $\partial_2 Zf(x,\omega) = -2\pi i(Z(Xf)(x,\omega) - x \cdot Zf(x,\omega))$.

We omit the technical proof here as it relies on standard density arguments ($C_c^\infty(\mathbb{R})$ being dense in $H^1(\mathbb{R})$) for one direction of the implication and known properties of absolutely continuous functions for the other.

III. Main result

The main ingredient of the proof of Theorem I.2 is the following proposition.

Proposition III.1. *Let* $\Lambda \subset \mathbb{R}^2$ *be a rational density lattice and let* $g \in \mathbb{H}^1(\mathbb{R})$ *be such that* (g,Λ) *is a Riesz basis for its closed linear span* $\mathcal{G}(g,\Lambda)$. *Then for any* $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$ *we have that* $-ag' + 2\pi ibXg \notin \mathcal{G}(g,\Lambda)$.

The proof of Proposition III.1 is technical and rather long. Here we will sketch the proof of a particular and much more straightforward case.

Proposition III.2. *Let* $\Lambda = \mathbb{Z} \times 2\mathbb{Z}$ *and let* $g \in \mathbb{H}^1(\mathbb{R})$ *be such that* (g,Λ) *is a Riesz basis for its closed linear span* $\mathcal{G}(g,\Lambda)$. *Then* $g' \notin \mathcal{G}(g,\Lambda)$.

We will denote the Zak transform of g by $G := Zg$ for simplicity. The condition of (g,Λ) being a Riesz basis for its span is then equivalent to [3, Lemma 2.3]

$$2A \le |G(x,\omega)|^2 + |G(x+\tfrac{1}{2},\omega)|^2 \le 2B, \quad a.e.(x,\omega) \in \mathbb{R}^2, \tag{2}$$

where $A, B > 0$ are the Riesz bounds.

Next we provide a characterization for a function being contained in the Gabor space.

Lemma III.3. *Let* g *and* $\Lambda = \mathbb{Z} \times 2\mathbb{Z}$ *be as above. Then* $f \in \mathcal{G}(g,\Lambda)$ *if and only if there exists some function* $h \in L^2_{loc}(\mathbb{R}^2)$ *that is* $\frac{1}{2}$*-periodic in* x *and* 1*-periodic in* ω *satisfying* $Zf(x,\omega) = h(x,\omega)G(x,\omega)$.

Sketch of proof. The condition $f \in \mathcal{G}(g,\Lambda)$ is equivalent to the existence of a sequence $(c_{m,n})_{m,n} \in \ell^2(\mathbb{Z}^2)$ such that

$$f = \sum_{m,n\in\mathbb{Z}} c_{m,n}\pi(m,2n)g.$$

The conclusion follows from the properties of the Zak transform and from the exponentials $e^{2\pi i(2nx+m\omega)}$ forming a Fourier basis for $L^2([0,\frac{1}{2}] \times [0,1])$. □

Sketch of proof of Proposition III.2. Let us assume that $g' \in \mathcal{G}(g,\Lambda)$. By Lemma III.3 and Lemma II.3, there exists a function h such that for a.e. $(x,\omega) \in \mathbb{R}^2$ we have $\partial_1 G(x,\omega) = h(x,\omega)G(x,\omega)$. This simple PDE leads to the general solution (after taking care of some measure theoretic issues)

$$G(x,\omega) = G(0,\omega)e^{\int_0^x h(s,\omega)ds}, \tag{3}$$

which holds for a.e. $(x,\omega) \in \mathbb{R}^2$.

Note now that as a consequence of (2) and (3), we obtain

$$0 < 2A \le |G(0,\omega)|^2 (e^{2\operatorname{Re}\int_0^x h(s,\omega)ds} + e^{2\operatorname{Re}\int_0^{x+\frac{1}{2}} h(s,\omega)ds})$$

which implies that for a.e. ω we have that $G(0,\omega) \neq 0$. Now using the quasiperiodicity of G, the periodicity of h and (3) we obtain

$$e^{2\pi i\omega}G(0,\omega)e^{\int_0^x h(s,\omega)ds} = G(0,\omega)e^{2\int_0^{\frac{1}{2}} h(s,\omega)ds}e^{\int_0^x h(s,\omega)ds},$$

which then reduces to $e^{2\int_0^{\frac{1}{2}} h(s,\omega)ds} = e^{2\pi i\omega}$. Therefore $G(x+\frac{1}{2},\omega) = \pm e^{\pi i\omega}G(x,\omega)$ and then $|G(x+\frac{1}{2},\omega)| =$

$|G(x,\omega)|$. Returning now to (2), we obtain $|G(x,\omega)|^2 \geq A$ for a.e. (x,ω), so the Zak transform of g is essentially bounded below. G is also bounded above as a consequence of (2). But this implies [1, Thm 3.1.d] that $(g, \mathbb{Z}\times\mathbb{Z})$ is a frame for $L^2(\mathbb{R})$. By Ron-Shen duality [8, Thm 7.4.3], $(g, \mathbb{Z}\times\mathbb{Z})$ must also be a Riesz sequence for its closed linear span, i.e., for $L^2(\mathbb{R})$, as the adjoint lattice of $\mathbb{Z}\times\mathbb{Z}$ is itself. This is a contradiction to the classical Balian-Low theorem as we have assumed from the beginning that $g \in \mathbb{H}^1(\mathbb{R})$. □

As stated before, the proof of Proposition III.1—the main result of [5]—is much more technically involved. The proof relies on showing that an entire line of additional time-frequency shifts is contained in $\mathfrak{I}(\mathcal{G}(g,\Lambda))$ using uniqueness properties of the solutions to a certain matrix differential equation that arises from the assumption of $-ag' + 2\pi ibXg \in \mathcal{G}(g,\Lambda)$ in the case of the lattice Λ being of the form $\frac{1}{Q}\mathbb{Z}\times P\mathbb{Z}$. This coupled with Theorem I.1 implies the entire line of additional time-frequency shifts is contained in Λ, which is absurd. From there standard metaplectic operator techniques [7] are used to generalize to arbitrary rational density lattices.

Sketch of the proof of Theorem I.2. The upper bound β is straightforward to estimate, even under weaker assumptions. Standard inequalities show that for any $g \in \mathbb{H}^1(\mathbb{R})$, any lattice $\Lambda \subset \mathbb{R}^2$ and any $z \in \mathbb{R}^2$

$$\operatorname{dist}(\pi(z)g, \mathcal{G}(g,\Lambda)) \leq \sqrt{\|g'\|_{L^2}^2 + \|2\pi iXg\|_{L^2}^2} \cdot \operatorname{dist}(z,\Lambda).$$

Let us denote by $\mathbb{P}$ the orthogonal projection onto $\mathcal{G}(g,\Lambda)$ in $L^2(\mathbb{R})$. Then $\operatorname{dist}(\pi(z)g, \mathcal{G}(g,\Lambda)) = \|(\operatorname{Id}-\mathbb{P})(\pi(z+\lambda)g)\|_{L^2}$ for any $\lambda \in \Lambda$ due to Lemma II.1. As a consequence of Proposition III.1, the $\mathbb{R}$-linear mapping $\mathbb{R}^2 \to L^2(\mathbb{R}), (a,b) \mapsto (\operatorname{Id}-\mathbb{P})(-ag' + 2\pi ibXg)$, with $L^2(\mathbb{R})$ viewed as an $\mathbb{R}$-linear space, is injective. Therefore, since $\mathbb{R}^2$ is finite dimensional, there exists some $c > 0$ such that

$$\|(\operatorname{Id}-\mathbb{P})(-ag' + 2\pi ibXg)\|_{L^2} \geq 2c\|(a,b)\|_2. \tag{4}$$

Now from Proposition II.2 we can write

$$\pi(a,b)g - g = -ag' + 2\pi ibXg + \varepsilon(a,b)$$

where the error term ε satisfies $\lim_{(a,b)\to(0,0)} \frac{\varepsilon(a,b)}{\|(a,b)\|_2} = 0$. Therefore there exists some $\delta > 0$ such that $\|\varepsilon(a,b)\|_{L^2} \leq c\|(a,b)\|_2$ for $\|(a,b)\| < \delta$. Since $(\operatorname{Id}-\mathbb{P})g = 0$, we obtain from (4) that $\operatorname{dist}(\pi(a,b)g, \mathcal{G}(g,\Lambda)) \geq c\|(a,b)\|_2$ for $\|(a,b)\|_2 < \delta$.

Finally, consider the compact subset obtained by removing the δ-balls centered at lattice points from the fundamental domain of Λ and denote it by C_δ. Clearly for any $z \in C_\delta$ we have that $\operatorname{dist}(z,\Lambda) \geq \delta$. Assume that $\operatorname{dist}(\pi(z)g, \mathcal{G}(g,\Lambda))$ is not bounded below on C_δ. Then we can find a sequence $(z_n)_n$ in C_δ such that $\operatorname{dist}(\pi(z_n)g, \mathcal{G}(g,\Lambda)) \to 0$. By compactness and possibly passing to a subsequence, we can assume that there exists $z^* \in C_\delta$ such that $z_n \to z^*$. But then $\operatorname{dist}(\pi(z^*)g, \mathcal{G}(g,\Lambda)) = 0$ and since $z^* \notin \Lambda$, this contradicts Theorem I.1. Therefore there exists $0 < c' \leq \inf_{z\in C_\delta} \operatorname{dist}(\pi(z)g, \mathcal{G}(g,\Lambda))$. We can conclude the proof by taking $\alpha = \min(c, \frac{\sqrt{2}c'}{\|A\|_{op}})$, where $\Lambda = A\mathbb{Z}^2$. □

IV. Final remarks

We conclude with two remarks. Firstly, under stronger assumptions on the window function, an explicit estimate for α from Theorem I.2 can be computed on a neighbourhood of the lattice points.

Remark IV.1. [5, Thm 5.4] Let $\Lambda \subset \mathbb{R}^2$ be a lattice (not necessarily of rational density) and let $g \in \mathbb{H}^1(\mathbb{R})$ be such that (g,Λ) is an orthonormal basis for its closed linear span $\mathcal{G}(g,\Lambda)$. Then there exists some $\varepsilon > 0$ such that

$$\operatorname{dist}(\pi(z)g, \mathcal{G}(g,\Lambda)) \geq \frac{\pi}{2\cdot\sqrt{\|g'\|_{L^2}^2 + \|2\pi iXg\|_{L^2}^2}} \operatorname{dist}(z,\Lambda)$$

for all z in an ε neighbourhood of the lattice points in Λ. If additionally $g, \hat{g} \in H^2(\mathbb{R})$, then ε can be chosen as $\varepsilon := \frac{\pi}{2\Delta\sqrt{\|g'\|_{L^2}^2 + \|2\pi iXg\|_{L^2}^2}}$, where $\Delta := 3\pi^2 \max(\|X^2g\|_{L^2}, \|\omega^2\hat{g}\|_{L^2}, \|Xg'\|_{L^2})$.

Secondly, the result of Proposition III.1 is a stronger version of the weak Balian-Low theorem in the case of $g \in \mathbb{H}^1(\mathbb{R})$ and Λ having rational density. In this setting, Proposition III.1 shows that no real linear combination of g' and iXg can be an element of $\mathcal{G}(g,\Lambda)$. The same argument can be made for the dual window $\tilde{g}$ which is also in $\mathbb{H}^1(\mathbb{R})$ and is also a generator for $\mathcal{G}(g,\Lambda)$.

Theorem IV.2 (Weak BLT)**.** *[9, Thm 8] Let $g \in L^2(\mathbb{R})$ and $\Lambda \subset \mathbb{R}^2$ be a lattice such that (g,Λ) is a Riesz basis for its closed linear span $\mathcal{G}$. Let $\tilde{g}$ be the dual window of g. Then at least one of $g', Xg, \tilde{g}', X\tilde{g}$ is not contained in $\mathcal{G}$.*

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