On the cosine operator function framework of windowed Shannon sampling operators

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***Abstract*—The aim of this paper is to consider the cosine-type Shannon sampling operators in the unified cosine operator function framework. In particular, we present the numerical estimates for the operator norms as well as for the order of approximation.**

I. Introduction

The aim of this paper is to study the windowed Shannon sampling operators

$$(B_{w,\boldsymbol{a}}f)(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{w}\right) s_{\boldsymbol{a}}(wt-k), \tag{1}$$

with the kernel function $s_{\boldsymbol{a}} \in L^1(\mathbb{R})$ for the signals $f \in C(\mathbb{R})$ (the space of uniformly continuous and bounded functions on $\mathbb{R}$ endowed with the supremum norm). The term "windowed" comes from the fact that we consider the kernel function

$$s_{\boldsymbol{a}}(t) = \int_0^1 \lambda_{\boldsymbol{a}}(u)\cos(\pi ut)du, \tag{2}$$

which is the CFT of a given even window function (see, e.g. [3]- [5], [12], [13])

$$\lambda_{\boldsymbol{a}}(u) = \sum_{k=0}^{m} a_k \cos(k\pi u),\ u \in [0,1], \tag{3}$$

where $\boldsymbol{a} = (a_0, ..., a_m) \in \mathbb{R}^{m+1}$, $m \geq 1$ and $\lambda_{\boldsymbol{a}}(u) = 0$ for $|u| \geq 1$. The operators (1) are well-defined if we assume for $\boldsymbol{a} \in \mathbb{R}^{m+1}$ that

$$\sum_{k=0}^{m} a_k = 1,\ \sum_{k=0}^{m}(-1)^k a_k = 0. \tag{4}$$

The operators (1), defined by the general kernel function, were introduced by P. L. Butzer and his school in 1977 (see [2] and literature therein); the approximation properties for (1), using specific kernel functions, were studied in [8]- [10].

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II. General approximation theorems

Let X be an arbitrary (real or complex) Banach space, and denote by $[X]$ the Banach algebra of all bounded linear operators U of X into itself. Next definitions and notations were introduced in [7].

Definition 1: (compare [11], [14]) A cosine operator function $C_h \in [X]$ $(h \geq 0)$ is defined by the properties:

1) $C_0 = I$ (identity operator),
2) $C_{h_1} \cdot C_{h_2} = \frac{1}{2}(C_{h_1+h_2} + C_{|h_1-h_2|})$,
3) $\|C_h f\| \leq T\|f\|$, the constant $T > 0$ is not dependent on $h > 0$.

Definition 2: The modulus of continuity of order $k \in \mathbb{N}$ is defined for $\delta \geq 0$ via the cosine operator function by

$$\omega_k(f,\delta) := \sup_{0 \leq h \leq \delta} \|(C_h - I)^k f\|. \tag{5}$$

Definition 3: The best approximation of $f \in X$ by elements of A_σ is defined by

$$E_\sigma(f) := \inf_{g_\sigma \in A_\sigma} \|f - g_\sigma\|.$$

In addition, let $\{A_\sigma\}_{\sigma>0}$ be a dense family of subspaces of X meaning that for the subspaces $A_\sigma \subset X$ with $A_{\sigma_1} \subset A_{\sigma_2}$, $0 < \sigma_1 < \sigma_2$, the union $\bigcup_{\sigma>0} A_\sigma$ is dense in X, i.e. for every $f \in X$ there exists a family $\{g_\sigma\}_{\sigma>0} \subset \bigcup_{\sigma>0} A_\sigma$ such that $\lim_{\sigma\to\infty} \|f - g_\sigma\| = 0$. Moreover, let $A_\sigma \subset X$ consist of the fixed points of a linear operator $S_\sigma : A_\sigma \to A_\sigma$, i.e. for any $g \in A_\sigma$ we have $S_\sigma g = g$.

The last situation is valid exactly for the Bernstein classes $\mathbb{B}^\infty_\sigma \subset C(\mathbb{R})$, consisting of the bounded functions on $\mathbb{R}$, which are entire functions $f(z)$ $(z \in \mathbb{C})$ of exponential type σ, i.e. $|f(z)| \leq e^{\sigma|y|}\|f\|_C \quad (z = x + iy \in \mathbb{C})$ (see, e.g. [6]).

Now the operator S_σ, mentioned above, is defined as the classical Whittaker-Kotel'nikov-Shannon operator

$$(S_w^{\mathrm{sinc}} g)(t) := \sum_{k=-\infty}^{\infty} g\left(\frac{k}{w}\right)\mathrm{sinc}(wt-k), \tag{6}$$

where $g \in \mathbb{B}^\infty_\sigma$, $\sigma < \pi w$ and $\mathrm{sinc}(t) := \frac{\sin \pi t}{\pi t}$. For $g \in \mathbb{B}^\infty_\sigma$, $\sigma < \pi w$, the equality

$$S_w^{\mathrm{sinc}} g = g$$

is valid due to the Whittaker-Kotel'nikov-Shannon theorem [6].

In the space $C(\mathbb{R})$, the natural cosine operator function $C_h \in [C(\mathbb{R})]$ is defined by the arithmetic mean of shifts of the function $f \in C(\mathbb{R})$

$$(C_h f)(x) = \frac{1}{2}\left(f(x+h) + f(x-h)\right), h \geq 0. \qquad (7)$$

In this case, the classical modulus of smoothness of order $2k$ ([7], cf. [1], p. 76) is defined by

$$\omega_{2k}(f, \delta) := \frac{1}{2^k} \sup_{0 \leq h \leq \delta} \|(C_h - I)^k f\|. \qquad (8)$$

We need to specify the best approximation of $f \in C(\mathbb{R})$ by $E_{\pi w-0}(f) := \inf\left\{\|f-g\| : g \in \mathbb{B}_\sigma^\infty, \sigma < \pi w\right\}$. A recent Jackson-type theorem ([17], Section 4.1, inequality (4.1); compare [16], Subsection 5.1.3) reads

$$E_{\pi w-0}(f) \leq M_k \omega_{2k}\left(f, 1/w\right), \qquad (9)$$

where we will use the cases $k = 1, 2, 3, 4$ only, with $M_1 = 5/8$, $M_2 = 65/216$, $M_3 = 0.11613...$, $M_4 = 0.03989...$, the constants close to the best possible ([17], Table 1 and Abstract).

In our previous paper [7], we developed a general cosine operator function framework to study the order of approximation for operators (1). The operators (1), $B_{w,\boldsymbol{a}} : C(\mathbb{R}) \to C(\mathbb{R})$, appear to be extensions of the operators $\widetilde{B}_{w,\boldsymbol{a}} : \mathbb{B}_\sigma^\infty \to C(\mathbb{R})\, (\sigma < \pi w)$, defined by

$$(\widetilde{B}_{w,\boldsymbol{a}} g)(t) := \sum_{j=0}^{m} a_j C_{j/w}\left(S_w^{\text{sinc}} g\right)(t), \qquad (10)$$

where the coefficients vector $\boldsymbol{a} = (a_0, ..., a_m) \in \mathbb{R}^{m+1}$ satisfies the equations (4). This extension from (10) to (1) becomes clear when in (10) we use (6) and (7) yielding

$$(\widetilde{B}_{w,\boldsymbol{a}} g)(t) = \sum_{k=-\infty}^{\infty} g\left(\frac{k}{w}\right) s_{\boldsymbol{a}}(wt - k),$$

where

$$s_{\boldsymbol{a}}(t) := \operatorname{sinc}(t) \sum_{j=0}^{m} (-1)^j a_j \frac{t^2}{t^2 - j^2}. \qquad (11)$$

Due to the condition (4), second equation, we can see by (11) that $s_{\boldsymbol{a}}(t) = O(t^{-3})$, hence the kernel function $s_{\boldsymbol{a}} \in L^1(\mathbb{R})$.

Now we announce two results from our general cosine operator function framework [7]. Following theorems generalize a classical direct approximation theorem for the Rogosinski operators (see, e.g., [1], Th. 2.4.8).

Theorem 1: Assume (4) are valid. Then for every $f \in C(\mathbb{R})$ the operator $B_{w,\boldsymbol{a}} : C(\mathbb{R}) \to C(\mathbb{R})$ has the order of approximation

$$\|B_{w,\boldsymbol{a}} f - f\| \leq \left(\|B_{w,\boldsymbol{a}}\|_{[C(\mathbb{R})]} + \sum_{k=0}^{m} |a_k|\right) E_{\pi w-0}(f) + \frac{1}{2}\omega_2\left(f, \frac{1}{w}\right) \sum_{l=1}^{m} l^2 |a_l|. \qquad (12)$$

Theorem 2: Let us fix the integer q with $m \geq q \geq 2$ and suppose that the coefficients $\boldsymbol{a} = (a_0, ..., a_m)$ satisfy the equalities (4) and

$$\sum_{k=p}^{m} k \binom{k+p-1}{2p-1} a_k = 0 \qquad (13)$$

for every $p = 1, 2, .., q-1$. Then for every $f \in C(\mathbb{R})$ we have

$$\|B_{w,\boldsymbol{a}} f - f\| \leq \left(\|B_{w,\boldsymbol{a}}\|_{[C(\mathbb{R})]} + \sum_{k=0}^{m} |a_k|\right) E_{\pi w-0}(f) + \frac{1}{2q}\omega_{2q}\left(f, \frac{1}{w}\right) \sum_{l=q}^{m} l|a_l| \binom{l+q-1}{2q-1}. \qquad (14)$$

III. Flat-Top windows and Shannon sampling operators

Nowadays, in the DFT methods the most popular window functions are the Nuttall [13] and the Flat-Top [4] windows. Both, in fact, are certain families of windows

$$w_j = \sum_{k=0}^{m} (-1)^k a_k \cos\left(\frac{2k\pi j}{N}\right), \; j = 0, 1, ..., N-1, \qquad (15)$$

including the Hann and the Blackman [5] windows known already a long time ago. The DFT methods use several sets of coefficients $\boldsymbol{a} = (a_0, ..., a_m)$, which are optimized for different aims like spectral leakage reduction ([3], Section 3.5.5; [12], Section 3.83.9), small sidelobes, etc.

In this section, we discuss which windows in the DFT methods satisfy the conditions (4) or (4) and (13). The windows (15) are defined in the nonsymmetric range $j = 0, ..., N-1$, but for the Shannon sampling operators we use the symmetric range, therefore the counterpart of (15) with $\boldsymbol{a} \in \mathbb{R}^5$ is the window function $\lambda_{\boldsymbol{a}}(u) = \sum_{k=0}^{4} a_k \cos(k\pi u)$. The solution of the system (4) is

$$\boldsymbol{a} = \left(\frac{1}{2} - c_1 - c_3; \frac{1}{2} - c_2; c_1; c_2; c_3\right) \in \mathbb{R}^5, \qquad (16)$$

where c_1, c_2, c_3 are arbitrary real parameters. We can rewrite the kernel $s_{\boldsymbol{a}}$ in (11) in the form

$$s_{\boldsymbol{a}}(t) = \frac{p_{\boldsymbol{a}}(t)\operatorname{sinc} t}{(16-t^2)(9-t^2)(4-t^2)(1-t^2)}, \qquad (17)$$

where $p_{\boldsymbol{a}}$ is an even polynomial of degree six. By Theorem 1, the coefficients (16) yield for the operators $B_{w,\boldsymbol{a}}$ the order of approximation estimated by $\omega_2\left(f, 1/w\right)$. However, the appropriate choice of parameters c_1, c_2, c_3 may be used to decrease the order of $s_{\boldsymbol{a}}(t)$ at infinity. If for $p_{\boldsymbol{a}}$ the coefficient at t^6 is equal to zero, then $s_{\boldsymbol{a}}(t) = O(t^{-5})$. Therefore, let us take $c_3 = 1/32 - c_1/4 + c_2/2$. Then

$$\boldsymbol{a} = \left(\frac{15}{32} - \frac{3c_1}{4} - \frac{c_2}{2}; \frac{1}{2} - c_2; c_1; c_2; \frac{1}{32} - \frac{c_1}{4} + \frac{c_2}{2}\right) \qquad (18)$$

and the even polynomial $p_{\boldsymbol{a}}$ is of degree four. In a similar way we can eliminate the powers t^4 and t^2 as well. If

$$\boldsymbol{a}=\left(\frac{35}{64}-\frac{5c_1}{4};\frac{21}{32}-c_1;c_1;c_1-\frac{5}{32};-\frac{3}{64}+\frac{c_1}{4}\right),$$

then $p_{\boldsymbol{a}}$ is an even quadratic polynomial and $s_{\boldsymbol{a}}(t)=O(t^{-7})$. Finally, we take $c_1=7/32$ and get $\boldsymbol{a}=(35,56,28,8,1)/128$, $p_{\boldsymbol{a}}(t)=315/2$ and $s_{\boldsymbol{a}}(t)=O(t^{-9})$.

By Theorem 2, the order of approximation for $B_{w,\boldsymbol{a}}$ is guaranteed via $\omega_4\left(f,1/w\right)$ if for the coefficients $\boldsymbol{a}\in\mathbb{R}^5$ equations (4) and (13) with $p=1$ are satisfied. In this case it follows

$$\boldsymbol{a}=\left(\frac{5}{8}+2c_1+3c_2;\frac{1}{2}-c_1;-\frac{1}{8}-2c_1-4c_2;c_1;c_2\right),$$

with an even polynomial $p_{\boldsymbol{a}}$ being of degree six and $s_{\boldsymbol{a}}(t)=O(t^{-3})$. In a similar manner the order $s_{\boldsymbol{a}}(t)=O(t^{-5})$ yields for

$$\boldsymbol{a}=\left(\frac{1}{2}+3c_2;\frac{9}{16};-4c_2;-\frac{1}{16};c_2\right).$$

The order $s_{\boldsymbol{a}}(t)=O(t^{-7})$ is guaranteed for

$$\boldsymbol{a}=\frac{1}{128}\left(55;72;12;-8;-3\right). \tag{19}$$

By Theorem 2, the order of approximation for $B_{w,\boldsymbol{a}}$ is guaranteed via $\omega_6\left(f,1/w\right)$ if for the coefficients $\boldsymbol{a}\in\mathbb{R}^5$ equations (4) and (13) with $p=1$ and $p=2$ are satisfied. In this case it follows

$$\boldsymbol{a}=\left(\frac{11}{16}-5c;\frac{15}{32}+4c;-\frac{3}{16}+4c;\frac{1}{32}-4c;c\right).$$

The order $s_{\boldsymbol{a}}(t)=O(t^{-5})$ is guaranteed for

$$\boldsymbol{a}=\frac{1}{128}(73;72;-12;-8;3). \tag{20}$$

Finally, we can manage the order of approximation $\omega_8\left(f,1/w\right)$ with $s_{\boldsymbol{a}}(t)=O(t^{-3})$ for

$$\boldsymbol{a}=\frac{1}{128}(93;56;-28;8;-1). \tag{21}$$

IV. Estimation of operator norms

In our previous papers ([8], [9], [10]) we calculated the exact values of certain operator norms $\|B_{w,\boldsymbol{a}}\|_{[C(\mathbb{R})]}$. Sometimes these calculations were very complicated, therefore we would like to have a simpler method for estimating the operator norms. Fortunately, for the cosine-sum windows (3) there is quite an easy way to do that. We illustrate the method with an example of the 5-term cosine-sum windows. We consider the coefficients $\boldsymbol{a}\in\mathbb{R}^5$, for which the conditions (4) with $m=4$ are fulfilled. In this case by (16) $\boldsymbol{a}=(1/2-c_1-c_3;1/2-c_2,c_1,c_2,c_3)\in\mathbb{R}^5$ for arbitrary $\boldsymbol{c}=(c_1,c_2,c_3,1)\in\mathbb{R}^4$. Denote $\mathbf{cos}(u):=(1,\cos\pi u,\cos 2\pi u,\cos 3\pi u,\cos 4\pi u)\in\mathbb{R}^5$. In given matrix (vector) notations the window function (3) has the form

$$\lambda_{\boldsymbol{a}}(u)=\mathbf{cos}(u)\,\boldsymbol{a}^T, \tag{22}$$

where

$$\boldsymbol{a}^T=\begin{pmatrix}-1&0&-1&1/2\\0&-1&0&1/2\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\1\end{pmatrix}\equiv A\boldsymbol{c}^T, \tag{23}$$

and $A\in\mathbb{R}^{5\times 4}$. Suppose we know the operator norms for four corresponding window functions $\lambda_{\boldsymbol{a}_j}(u)=\mathbf{cos}(u)\,\boldsymbol{a}_j^T,\ j=1,2,3,4$, with $\boldsymbol{a}_j=(1/2-c_{1j}-c_{3j};1/2-c_{2j},c_{1j},c_{2j},c_{3j})$ and $\boldsymbol{c}_j=(c_{1j},c_{2j},c_{3j},1)$. By (23),

$$\lambda_{\boldsymbol{a}_j}(u)=\mathbf{cos}(u)\left(A\boldsymbol{c}_j^T\right). \tag{24}$$

Denote $\boldsymbol{d}=(d_1,d_2,d_3,d_4)\in\mathbb{R}^4$ and suppose that the following equation

$$C\boldsymbol{d}^T\equiv\begin{pmatrix}c_{11}&c_{12}&c_{13}&c_{14}\\c_{21}&c_{22}&c_{23}&c_{24}\\c_{31}&c_{32}&c_{33}&c_{34}\\1&1&1&1\end{pmatrix}\boldsymbol{d}^T=\boldsymbol{c}^T \tag{25}$$

has a unique solution, i.e. the matrix C is invertible. Therefore, $\boldsymbol{d}^T=C^{-1}\boldsymbol{c}^T$, and we claim that

$$\lambda_{\boldsymbol{a}}(u)=\sum_{j=1}^{4}d_j\lambda_{\boldsymbol{a}_j}(u). \tag{26}$$

Indeed, by equations (23) and (25), we have

$$\boldsymbol{a}^T=A\boldsymbol{c}^T=A\left(C\boldsymbol{d}^T\right)=A\left(\sum_{j=1}^{4}d_j\boldsymbol{c}_j^T\right)=\sum_{j=1}^{4}d_jA\boldsymbol{c}_j^T.$$

Now by (24) for the corresponding window functions it follows

$$\lambda_{\boldsymbol{a}}(u)=\mathbf{cos}(u)\boldsymbol{a}^T=\sum_{j=1}^{4}d_j\mathbf{cos}(u)\left(A\boldsymbol{c}_j^T\right)=\sum_{j=1}^{4}d_j\lambda_{\boldsymbol{a}_j}(u).$$

The last equation yields $B_{w,\boldsymbol{a}}f=\sum\limits_{j=1}^{4}d_jB_{w,\boldsymbol{a}_j}f$ and for the operator norms

$$\|B_{w,\boldsymbol{a}}\|_{[C(\mathbb{R})]}\leq\sum_{j=1}^{4}|d_j|\left\|B_{w,\boldsymbol{a}_j}\right\|_{[C(\mathbb{R})]}. \tag{27}$$

Since the values of our known operator norms are quite close to 1, the estimate (27) is heavily dependent on the l_1 norm of the vector $\boldsymbol{d}$. In Table 1 we collected several operators with known operator norms. The set of these norms and the corresponding parameter vectors $\boldsymbol{a}\in\mathbb{R}^5$ constitutes a database for estimating the operator norms via (27). We wrote a MATLAB routine which uses this database to solve the equations of type (25) and finds minimal value of $|\boldsymbol{d}|_1$ for the corresponding unknown operator norm. The results are presented in the next Section.

V. Applications

Table I summarizes data about operators with known operator norms. The last column presents estimates of constants in Theorems 1 and 2. The index k in M_k indicates that the corresponding order of approximation is measured via the modulus of continuity $\omega_{2k}(f, 1/w)$, i.e. of order $2k$. Table II presents data about approximations by operators whose operator norms have been estimated by the MATLAB routine.

TABLE I

Operator name and parameter vector $\boldsymbol{a}$	Kernel $s_{\boldsymbol{a}}(t)$ or order at infinity	Norms $\|B_{w,\boldsymbol{a}}\|_{[C(\mathbb{R})]}$ and order of approximation $\|B_{w,\boldsymbol{a}}f - f\| \leq M_k(\boldsymbol{a})\omega_{2k}\left(f, \frac{1}{w}\right)$
1. Hann [8], [5] $(1/2; 1/2; 0; 0; 0)$	$\frac{\mathrm{sinc}(t)}{2(1-t^2)}$	$\frac{10}{3\pi} = 1.0610...$; $M_1(\boldsymbol{a}) \leq 1.538...$
2a. Blackman [9], [5], close to "exact" Blackman $(27/64; 1/2; 5/64; 0; 0)$	$\frac{3(9-t^2)\,\mathrm{sinc}(t)}{16(1-t^2)(4-t^2)}$	$\frac{3973}{1260\pi} = 1.0036...$; $M_1(\boldsymbol{a}) \leq 1.660...$
2b. Blackman [9] $(3/8; 1/2; 1/8; 0; 0)$	$\frac{3\,\mathrm{sinc}(t)}{2(1-t^2)(4-t^2)}$	$\frac{332}{105\pi} = 1.0064...$; $M_1(\boldsymbol{a}) \leq 1.754...$
2c. Blackman [9] $(0; 1/2; 1/2, 0; 0)$	$\frac{3t^2\,\mathrm{sinc}(t)}{2(1-t^2)(4-t^2)}$	$\frac{362}{105\pi} = 1.0974...$; $M_1(\boldsymbol{a}) \leq 2.560...$
3a. Blackman-Harris [15] $(0; 0; 1/2; 1/2; 0)$	$= O\left(t^{-3}\right)$	$\frac{1802}{495\pi} = 1.1587...$; $M_1(\boldsymbol{a}) \leq 4.599...$
3b. [10], Th.6; $(48; 63; 16; 1; 0)\,/128$	$= O\left(t^{-3}\right)$	$\frac{43577}{13860\pi} = 1.0007...$; $M_1(\boldsymbol{a}) \leq 1.781...$
3c. [10], Th.6; $(63; 48; 1; 16; 0)\,/128$	$= O\left(t^{-3}\right)$	$\frac{91459}{27720\pi} = 1.0502...$; $M_1(\boldsymbol{a}) \leq 2.047...$
3d. [10], Th.6; $(1; 1; 15; 15; 0)\,/32$	$= O\left(t^{-3}\right)$	$\frac{2671}{770\pi} = 1.1041...$; $M_1(\boldsymbol{a}) \leq 4.377...$
3e. [8], Th.1; $(10; 15; 6; 1; 0)\,/32$	$= O\left(t^{-3}\right)$	$\frac{3632}{1155\pi} = 1.0009...$; $M_1(\boldsymbol{a}) \leq 2.000...$
4a. [8], Th.1; $(35; 56; 28; 8; 1)\,/128$	$= O\left(t^{-9}\right)$	$\frac{141536}{45045\pi} = 1.0001...$; $M_1(\boldsymbol{a}) \leq 2.250...$

TABLE II

Operator name and parameter vector $\boldsymbol{a}$	Kernel $s_{\boldsymbol{a}}(t)$ or order at infinity	Estimates of norms $\|B_{w,\boldsymbol{a}}\|_{[C(\mathbb{R})]}$ and order of approximation $\|B_{w,\boldsymbol{a}}f - f\| \leq M_k(\boldsymbol{a})\omega_{2k}\left(f, \frac{1}{w}\right)$
2d. Blackman [9]; $(5/8, 1/2, -1/8; 0; 0)$	$\frac{\left(5/2-t^2\right)\mathrm{sinc}(t)}{(1-t^2)(4-t^2)}$	$\leq 1.6006...$; $M_2(\boldsymbol{a}) \leq 0.920...$
3f. Nuttall [13]; $(0.355768; 0487396; 0.144232; 0.012604; 0)$	$= O\left(t^{-3}\right)$	$\leq 1.1608...$; $M_1(\boldsymbol{a}) \leq 1.939...$
3g. Blackman-Harris [15]; $(22; 15; -6; 1; 0)\,/32$	$= O\left(t^{-3}\right)$	$\leq 2.3445...$; $M_3(\boldsymbol{a}) \leq 3.580...$
3h. Blackman-Harris [7]; $(1/2; 9/16; 0, -1/16; 0)$	$\frac{9\,\mathrm{sinc}(t)}{2(1-t^2)(9-t^2)}$	$\leq 1.5292...$; $M_1(\boldsymbol{a}) \leq 4.057...$
4b. Section III (20); $(73; 72; -12; -8; 3)\,/128$	$= O\left(t^{-5}\right)$	$\leq 5.1261...$; $M_3(\boldsymbol{a}) \leq 0.836...$
4c. Section III (19); $(55; 72; 12; -8; -3)\,/128$	$= O\left(t^{-7}\right)$	$\leq 7.0043...$; $M_2(\boldsymbol{a}) \leq 2.877...$
4d. Section III (21); $(93; 56; -28; 8; -1)\,/128$	$= O\left(t^{-3}\right)$	$\leq 3.1451...$; $M_4(\boldsymbol{a}) \leq 0.169...$

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