Analysis of Shearlet Coorbit Spaces in arbitrary Dimensions using Coarse Geometry

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***Abstract*—In order to analyze anisotropic information of signals, the shearlet transform has been introduced as class of directionally selective wavelet transforms. One way of describing the approximation-theoretic properties of such generalized wavelet systems relies on *coorbit spaces*, i.e., spaces defined in terms of sparsity properties with respect to the system. In higher dimensions, there are several distinct possibilities for the definition of shearlet systems, and their approximation-theoretic properties are currently not well-understood.**

In this note, we investigate shearlet systems in higher dimensions derived from two particular classes of shearlet groups, the standard shearlet group and the Toeplitz shearlet group. We want to show that different groups lead to different approximation theories. The analysis of the associated coorbit spaces relies on an alternative description via *decomposition spaces* that was recently established.

For a shearlet group, this identification is based on a covering of the associated dual orbit induced by the shearlet group. The geometry of the sets in this covering is the determining factor for the associated decomposition space. We will see that the orbit can be equipped with a metric structure that encodes essential properties of this covering. The orbit map then allows to compare the geometric properties of coverings induced by different groups without the need to explicitly compute the respective coverings, which gets increasingly difficult for higher dimensions.

This argument relies on a rigidity theorem which states that *geometrically incompatible* coverings lead to different decomposition spaces in almost all cases.

I. Introduction

The original *shearlet group* G was introduced in [1] as the semidirect product $\mathbb{R}^2 \rtimes H$, where

$$H = \left\{ \pm \begin{pmatrix} a & b \\ 0 & a^{1/2} \end{pmatrix} : a > 0, b \in \mathbb{R} \right\}.$$

G acts by translations and dilations on the Hilbert space $\mathrm{L}^2(\mathbb{R}^2)$. This action, applied to a suitably chosen mother shearlet, gives rise to a shearlet system, and an associated shearlet transform.

The main motivation for introducing shearlet systems was that the anisotropic scaling inherent in the dilation group gave rise to shearlet systems whose approximation-theoretic properties improved significantly on the classical wavelets based on isotropic scaling. It was shown in [1] that the coorbit theory of Feichtinger and Gröchenig also applies to the shearlet group, thus allowing to consistently define smoothness spaces in terms of their shearlet coefficient decay.

Extensions to higher dimensions soon followed [2], retaining the features of the two-dimensional examples; specifically, coorbit theory also applies to the higher-dimensional shearlet systems. However, through [3] and subsequently [4], [5], it became clear that as the dimension increases, the number of shearlet groups that may be employed rapidly grows. In this note, we are particularly interested in the standard shearlet groups H^λ defined by

$$\left\{ \epsilon \begin{pmatrix} a & b_1 & \dots & b_{d-1} \\ & a^{\lambda_1} & 0 \dots & 0 \\ & & \ddots & 0 \\ & & & a^{\lambda_{d-1}} \end{pmatrix} \middle| \begin{array}{l} a > 0, \\ b_i \in \mathbb{R}, \\ \epsilon \in \{\pm 1\} \end{array} \right\}$$

for $\lambda = (\lambda_1, \dots, \lambda_{d-1}) \in \mathbb{R}^{d-1}$ (cf. Example 17 in [5]) and the Toeplitz shearlet groups H^δ defined by

$$\left\{ \epsilon \begin{pmatrix} a & b_1 & b_2 & \dots & b_{d-1} \\ & a^{1-\delta} & b_1 a^{-\delta} & \dots & b_{d-2} a^{-\delta} \\ & & \ddots & \ddots & \\ & & & \ddots & b_1 a^{-(d-2)\delta} \\ & & & & a^{1-(d-1)\delta} \end{pmatrix} \middle| \begin{array}{l} a > 0, \\ b_i \in \mathbb{R}, \\ \epsilon \in \{\pm 1\} \end{array} \right\}$$

for $\delta \in \mathbb{R}$ in $d \geq 2$ dimensions (cf. Example 18 in [5]).

It is not obvious whether two different groups taken from these families necessarily define different coorbit spaces; there exist wavelet systems arising from different dilation groups that have the same coorbit spaces up to suitable identification ([6]). In this note, we completely clarify this question, using recent results from the theory of decomposition spaces.

Section II contains the basic definitions regarding generalized wavelet transforms. We then introduce in Section III and IV coorbit spaces and decomposition spaces and explain how the former can be viewed as a special case of the latter. This identification allows us to employ decomposition space methods in order to compare coorbit spaces associated to different shearlet groups. This comparison is based on a rigidity result whose application in our setting boils down to the study of certain coverings. In Section V, we equip the orbit and the group with metrics that encode geometric information about the relation of the associated coverings and in Section

VI we apply a coarse geometric viewpoint, showing that each shearlet dilation group of these two classes gives rise to a different scale of coorbit spaces.

II. Wavelet Transforms in Higher Dimensions

For a closed matrix group $H < \mathrm{GL}(\mathbb{R}^d)$ let $G := \mathbb{R}^d \rtimes H$ be the group of affine mappings generated by H and translations. The group multiplication in G is given by $(x,h)(y,g) = (x+hy, hg)$ and a left Haar measure on G is $\mathrm{d}(x,h) = |\det(h)|^{-1}\mathrm{d}x\mathrm{d}h$, where $\mathrm{d}h$ denotes a left Haar measure on H.

The group G acts unitarily on $\mathrm{L}^2(\mathbb{R}^d)$ through the *quasi-regular representation*

$$[\pi(x,h)f](y) = |\det(h)|^{-1/2} f\left(h^{-1}(y-x)\right)$$

for $f \in \mathrm{L}^2(\mathbb{R}^d)$. In the following, we assume that H is chosen from the above list of subgroups of $\mathrm{GL}(\mathbb{R}^d)$. As a consequence (see e.g. [4]), π is irreducible and square-integrable, in which case we call the group H *admissible*. Admissibility of H is connected to the existence of some $\xi \in \mathbb{R}^d$ such that the *dual orbit* $H^{-T}\xi =: \mathcal{O} \subset \mathbb{R}^d$ is open with a complement of measure 0 (cf. [6]).

Then, after choosing a wavelet $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ the associated wavelet transform of $f \in \mathrm{L}^2(\mathbb{R}^d)$ is defined as

$$\mathcal{W}_\psi f : (x,h) \mapsto \langle f, \pi(x,h)\psi \rangle.$$

For a suitable ψ, the mapping $f \mapsto \mathcal{W}_\psi f$ is a scalar multiple of an isometry from $\mathrm{L}^2(\mathbb{R}^d)$ to $\mathrm{L}^2(G)$, which gives rise to the (weak-sense) inversion formula

$$f = \frac{1}{C_\psi} \int_G \mathcal{W}_\psi f(x,h)\pi(x,h)\psi \mathrm{d}(x,h) \ ,$$

for some $C_\psi > 0$, i.e., each $f \in \mathrm{L}^2(\mathbb{R}^d)$ is a continuous superposition of the wavelet system.

III. Coorbit Spaces

Coorbit spaces are defined in terms of decay behaviour of the generalized wavelet transform. To give a precise definition, we introduce weighted mixed L^p-spaces on G, denoted by $\mathrm{L}^{p,q}_v(G)$. By definition, this space is the set of functions

$$\left\{ f : G \to \mathbb{C} : \|f\|_{\mathrm{L}^{p,q}_v} < \infty \right\},$$

where

$$\|f\|_{\mathrm{L}^{p,q}_v} := \int_H \left(\int_{\mathbb{R}^3} |f(x,h)|^p \, v(x,h)^p \mathrm{d}x \right)^{q/p} \frac{\mathrm{d}h}{|\det(h)|}.$$

This definition is valid for $0 < p,q < \infty$, for $p = \infty$ or $q = \infty$ the essential supremum has to be taken at the appropriate place instead. The function $v : G \to \mathbb{R}^{>0}$ is a weight function that fulfills the condition $v(ghk) \le v_0(g)v(h)v_0(k)$ for some submultiplicative weight v_0. Thus the expression $\|W_\psi f\|_{\mathrm{L}^{p,q}_v}$ can be read as a measure of wavelet coefficient decay of f. We will exclusively consider weights which only depend on H. The coorbit space $\mathrm{Co}\left(\mathrm{L}^{p,q}_v(\mathbb{R}^d \rtimes H)\right)$ is then defined as the space

$$\left\{ f \in (\mathcal{H}^1_w)^\neg : W_\psi f \in W(\mathrm{L}^{p,q}_v(G)) \right\} \tag{1}$$

for some suitable wavelet ψ and a control weight w of $\mathrm{L}^{p,q}_v(G)$ on H in the sense of [7] (4.10). The space $(\mathcal{H}^1_w)^\neg$ denotes the space of antilinear functionals on

$$\mathcal{H}^1_w := \left\{ f \in \mathrm{L}^2(\mathbb{R}^d) : W_\psi f \in \mathrm{L}^1_w(G) \right\}$$

and $W(Y)$ for a function space Y on G denotes the Wiener amalgam space defined by $W_Q(Y) := \{ f \in \mathrm{L}^\infty_{\mathrm{loc}}(G) | M_Q f \in Y\}$ with quasi-norm $\|f\|_{W_Q(Y)} := \|M_Q f\|_Y$ for $f \in W_Q(Y)$, where the *maximal function* $M_Q f$ for some suitable unit neighborhood $Q \subset G$ is $M_Q f : G \to [0,\infty], \ x \mapsto \operatorname{ess\,sup}_{y \in xQ} |f(y)|$.

The appearance of the Wiener amalgam space in (1) is necessary to guarantee consistently defined quasi-Banach spaces in the case $\{p,q\} \cap (0,1) \neq \emptyset$, see [8] and [9]. In the classical coorbit theory for Banach spaces, which was developed in [7], [10], the Wiener amalgam space is replaced by $\mathrm{L}^{p,q}_v(G)$ and this change leads to the same space for $p, q \ge 1$, see [8].

Many useful properties of these spaces are known and hold in the quasi-Banach space case as well as in the Banach space case. The most prominent examples of coorbit spaces associated to generalized wavelet transforms are the homogeneous Besov spaces and the modulation spaces. However, each shearlet group gives rise to its scale of coorbit spaces, as well; see [3],[11], and [12].

IV. Decomposition Spaces

The starting point for the definition of decomposition spaces is the notion of an *admissible covering* $\mathcal{Q} = (Q_i)_{i\in I}$ of some open set $\mathcal{O} \subset \mathbb{R}^d$ (cf. [13]) which is a family of nonempty sets such that

i) $\bigcup_{i\in I} Q_i = \mathcal{O}$ and
ii) $\sup_{i\in I} |\{ j \in I : Q_i \cap Q_j \neq \emptyset \}| < \infty$,

where $|A|$ denotes the number of elements in the set A.

The main tool for the localization is a special partition of unity $\Phi = (\varphi_i)_{i\in I}$ subordinate to $\mathcal{Q}$, also called L^p-BAPU (bounded admissible partition of unity), with the following properties

i) $\varphi_i \in C^\infty_c(\mathcal{O}) \quad \forall i \in I$,
ii) $\sum_{i\in I} \varphi_i(x) = 1 \quad \forall x \in \mathcal{O}$,
iii) $\varphi_i(x) = 0$ for $x \in \mathbb{R}^d \setminus Q_i$ and $i \in I$,
iv) if $1 \le p \le \infty$: $\sup_{i\in I} \|\mathcal{F}^{-1}\varphi_i\|_{\mathrm{L}^1} < \infty$
 if $0 < p < 1$: $\sup_{i\in I} |\det(T_i)|^{\frac{1}{p}-1} \|\mathcal{F}^{-1}\varphi_i\|_{\mathrm{L}^p} < \infty$,

where we have to further assume in the case $0 < p < 1$ that the covering $\mathcal{Q}$ has the structure $Q_i = T_i Q + b_i$ with $T_i \in \mathrm{GL}(\mathbb{R}^d)$, $b_i \in \mathbb{R}^d$ and an open, precompact set Q ($\mathcal{Q}$ is then called a *structured admissible covering*). The definition of decomposition spaces requires one last ingredient, namely a weight $(u_i)_{i\in I}$ such that there exists $C > 0$ with $u_i \le C u_j$ for all $i, j \in I : Q_i \cap Q_j \neq \emptyset$, a weight with this property is also called $\mathcal{Q}$-*moderate*. The interpretation of this property is that the value of $(u_i)_{i\in I}$ is comparable for indices corresponding to sets which are "close" to each other. Finally, we define the decomposition space with respect to the covering $\mathcal{Q}$ and the weight $(u_i)_{i\in I}$ with integrability exponents $0 < p, q \le \infty$ as

$$\mathcal{D}(\mathcal{Q}, \mathrm{L}^p, \ell^q_u) := \{ f \in \mathcal{D}'(\mathcal{O}) : \|f\|_{\mathcal{D}(\mathcal{Q},\mathrm{L}^p,\ell^q_u)} < \infty \}$$

for

$$\|f\|_{\mathcal{D}(\mathcal{Q},\mathrm{L}^p,\ell^q_u)} := \left\| \left(u_i \cdot \|\mathcal{F}^{-1}(\varphi_i f)\|_{\mathrm{L}^p(\mathbb{R}^d)}\right)_{i\in I} \right\|_{\ell^q(I)} .$$

As the notation suggests, the decomposition spaces are independent of the precise choice of Φ ([9] Corollary 3.4.11).

In order to describe coorbit spaces as decomposition spaces, we need to associate a covering of the frequencies to a given dilation group. This is done using the *dual action*

$$H \times \mathbb{R}^d \ni (h,\xi) \mapsto h^{-T}\xi \ .$$

In the case of the shearlet groups studied here, the set $\mathcal{O} = \mathbb{R}^* \times \mathbb{R}^{d-1}$ is the orbit of the dual action, on which H acts freely. I.e., for $\xi_0 = (1,0,\dots,0)^T$, the orbit map $p_{\xi_0} : H \to \mathcal{O}, h \mapsto h^{-T}\xi_0$ is bijective. We then pick a *well-spread* family in H, i.e. a family of elements $(h_i)_{i\in I}$ with the properties

i) there exists a relatively compact neighborhood $U \subset H$ of the identity such that $\bigcup_{i\in I} h_i U = H$ and
ii) there exists a neighborhood $V \subset H$ of the identity such that $h_i V \cap h_j V = \emptyset$ for $i \neq j$.

The *dual covering induced by* H is then given by the family $\mathcal{Q} = (Q_i)_{i\in I}$, where $Q_i = p_{\xi_0}(h_i U)$. It can be shown that well-spread families always exist, and that the induced covering is indeed an admissible covering in the sense of decomposition space theory, for which L^p-BAPUs exist [9]. Furthermore, there always exist induced coverings consisting of open and connected sets, which we call *induced connected coverings*, see [14] Corollary 2.5.9.

There always exists a discretization of the weight v, i.e. an associated weight on the index set I, which enables a decomposition space description of the coorbit space.

Theorem 4.1 ([9] Theorem 4.6.3): Let $\mathcal{Q}$ be a covering of the dual orbit $\mathcal{O}$ induced by H, $0 < p,q \leq \infty$ and $u = (u_i)_{i\in I}$ a suitable (discrete) weight on I associated to v, then the Fourier transform $\mathcal{F} : \mathrm{Co}\left(\mathrm{L}^{p,q}_v(\mathbb{R}^d \rtimes H)\right) \to \mathcal{D}(\mathcal{Q},\mathrm{L}^p,\ell^q_u)$ is an isomorphism of (quasi-) Banach spaces.

Definition 4.2: We call two coorbit spaces *equivalent* if they are isomorphic to the same decomposition space via $\mathcal{F}$.

In the last part of this section, we state a result that gives necessary conditions for the equality of two decomposition spaces in terms of their ingredients $(p,q,u,\mathcal{Q})$. First, we have to introduce for two coverings $\mathcal{P} = (P_j)_{j\in J}$ and $\mathcal{Q} = (Q_i)_{i\in I}$ of the same set $\mathcal{O}$ and $i \in I, j \in J$ the notion of *intersection sets*, given by $I_j := \{i \in I : Q_i \cap P_j \neq \emptyset\}$ and $J_i := \{j \in J : Q_i \cap P_j \neq \emptyset\}$.

Definition 4.3: With the notation above, we say that the coverings are *weakly equivalent* if $\sup_{j\in J} |I_j| < \infty$ and $\sup_{i\in I} |J_i| < \infty$.

Now we can formulate the following rigidity result.

Theorem 4.4 ([15] Theorem 6.9): Let $\mathcal{O}$ be an open set, $0 < p_1, p_2, q_1, q_2 \leq \infty$ and assume that $\mathcal{Q} = (Q_i)_{i\in I}$ and $\mathcal{P} = (P_j)_{j\in J}$ are admissible coverings consisting of open sets of $\mathcal{O}$ for which there exist an L^{p_1}-BAPU and an L^{p_2}-BAPU, respectively. Furthermore, let $u = (u_i)_{i\in I}$ be $\mathcal{Q}$-moderate and $\tilde{u} = (\tilde{u}_j)_{j\in J}$ be $\mathcal{P}$-moderate weights. If

$$\mathcal{D}\left(\mathcal{Q},\mathrm{L}^{p_1},\ell^{q_1}_u(I)\right) = \mathcal{D}\left(\mathcal{P},\mathrm{L}^{p_2},\ell^{q_2}_{\tilde{u}}(J)\right) \qquad (2)$$

holds with equivalent (quasi-) norms, then we have

i) $p_1 = p_2$ and $q_1 = q_2$,
ii) in the case $(p_1,q_1) \neq (2,2)$ the coverings $\mathcal{P},\mathcal{Q}$ are weakly equivalent.

The pertinent condition in our setting is ii) because it allows to conclude that two decomposition spaces are nonequivalent by studying the relation of the induced coverings of their associated groups. This theorem has a partial converse (cf. Theorem 6.11 [15]).

V. Coarse Geometric Structure on dual Orbit and dilation group

In contrast to topology, where the small scale properties of a space are decisive, coarse geometry, also called coarse topology, puts special focus on the large-scale properties. It turns out that this perspective facilitates the study of weak equivalence of induced coverings. We say a map $f : (X,d_X) \to (Y,d_Y)$ between metric spaces is a *quasi-isometry* ([16]) if there exist $a,b > 0$ such that

$$b^{-1} d_X(x,y) - a \leq d_Y(f(x),f(y)) \leq b d_X(x,y) + a$$

for all $x,y \in X$. Quasi-isometric maps preserve the coarse structure of a metric space.

Let $\mathcal{Q} = (Q_i)_{i\in I}$ be a covering of the dual orbit $\mathcal{O}$. For $x,y \in \mathcal{O}$, we say x and y are connected by a *$\mathcal{Q}$-chain (of length m)* if there exist $Q_1,\dots,Q_m \in \mathcal{Q}$ such that $x \in Q_1$, $y \in Q_m$ and $Q_k \cap Q_{k+1} \neq \emptyset$ for all $k \in \{1,\dots,m-1\}$ ([14] Definition 5.2.1). This allows us to define a metric on $\mathcal{O}$ associated to a covering via

$$d_{\mathcal{Q}}(x,y) = \inf\left\{ m \in \mathbb{N} \;\middle|\; \begin{array}{l} x,y \text{ are connected by a} \\ \mathcal{Q}\text{-chain of length } m \end{array} \right\}$$

for $x \neq y$ and $d_{\mathcal{Q}}(x,x) = 0$ ([14] Definition 5.2.2). Weak equivalence of coverings of the dual orbit can be checked by investigating whether a specific map is a quasi-isometry, as was already remarked by Feichtinger and Gröbner in [13] Theorem 3.8 in a similar setting.

Theorem 5.1 ([14] Theorem 5.2.6): Let $\mathcal{Q}$ and $\mathcal{P}$ be induced connected coverings of $\mathcal{O}$. The following statements are equivalent:

i) The coverings $\mathcal{Q}$ and $\mathcal{P}$ are weakly equivalent.
ii) The map $\mathrm{id}^{\mathcal{P}}_{\mathcal{Q}}(\mathcal{O},d_{\mathcal{Q}}) \to (\mathcal{O},d_{\mathcal{P}}),\ x \mapsto x$ is a quasi-isometry.

Weak equivalence of induced coverings can be connected to the study of metric properties of a certain map between the inducing groups. To this end, we equip a group H with a variant of the word metric associated to a unit neighborhood $W \subset H$ defined by $d_W(x,y) = \inf\left\{m \in \mathbb{N} | x^{-1}y \in W^m\right\}$ for $x \neq y$ and $d_W(x,x) := 0$. In the next section, we show how to gain insights from this metric through Theorem 6.3.

VI. Comparison of Shearlet Coorbit Spaces in higher dimensions

In this section, let $H_1, H_2 \in \{H^\lambda\} \cup \{H^\delta\} \subset \mathbb{R}^d$ with $H_1 \neq H_2$. Let $W \subset H_1$ and $V \subset H_2$ be two relatively compact, connected, symmetric unit neighborhoods. Furthermore, let $\mathcal{Q} = (h_i^{-T}Q)_{i\in I}$ and $\mathcal{P} = (g_j^{-T}P)_{j\in J}$ be two induced connected coverings by H_1 and H_2, respectively. For some $\xi_1 \in Q$ and $\xi_2 \in P$, denote the associated orbit maps by

$$p_{\xi_1}^{H_1} : (H_1, d_W) \to (\mathcal{O}, d_{\mathcal{Q}}), h \mapsto h^{-T}\xi_1$$
$$p_{\xi_2}^{H_2} : (H_2, d_V) \to (\mathcal{O}, d_{\mathcal{P}}), h \mapsto h^{-T}\xi_2.$$

Lastly, define the map $\mathrm{id}_{\mathcal{Q}}^{\mathcal{P}} : (\mathcal{O}, d_{\mathcal{Q}}) \to (\mathcal{O}, d_{\mathcal{P}}), x \mapsto x.$

Definition 6.1 ([14] Definition 2.6.10): We call H_1 and H_2 *coorbit equivalent* if all induced coverings of these groups are weakly equivalent.

The mentioned converse of Theorem 4.4 implies that coorbit equivalent groups give rise to equivalent coorbit spaces. In order to make use of the coarse geometric viewpoint, the following feature of the orbit map is pivotal.

Theorem 6.2 ([14] Theorem 5.4.12): The orbit maps $p_{\xi_1}^{H_1}, p_{\xi_2}^{H_2}$ are quasi-isometries.

Denote by $\left(p_{\xi_2}^{H_2}\right)^{-1}$ an arbitrary right inverse of $p_{\xi_2}^{H_2}$ (there exists at least one since $p_{\xi_2}^{H_2}$ is surjective). As an inverse of a quasi-isometry, the map $\left(p_{\xi_2}^{H_2}\right)^{-1}$ is itself a quasi-isometry. With these preparations, we can state the main theorem of this section.

Theorem 6.3 ([14] Theorem 5.4.13): The following statements are equivalent:

i) H_1 and H_2 are coorbit equivalent.
ii) The map $\phi := \left(p_{\xi_2}^{H_2}\right)^{-1} \circ \mathrm{id}_{\mathcal{Q}}^{\mathcal{P}} \circ p_{\xi_1}^{H_1} : (H_1, d_W) \to (H_2, d_V)$ is a quasi-isometry.

We can apply Theorem 6.3 in the following way to the distinct shearlet Groups H_1 and H_2:

1) Take relatively compact, connected, symmetric unit neighborhoods $W \subset H_1, V \subset H_2$ with the property that certain left shifts of W and V tile H_1 and H_2, respectively.
2) Determine sequences $(h_n)_{n\in\mathbb{N}}$ and $(h'_n)_{n\in\mathbb{N}}$ in H_1 such that $h_n^{-1}h'_n \in W$ for all $n \in \mathbb{N}$ but $\left(\phi(h_n)^{-1}\phi(h'_n)\right)_{n\in\mathbb{N}}$ is not relatively compact in H_2. One way to establish this property is by showing that one entry in the matrix $\left(\phi(h_n)^{-1}\phi(h'_n)\right)_{n\in\mathbb{N}}$ is unbounded. Note, that we do not need to know all entries of the matrices $\phi(h_n)^{-1}$ and $\phi(h'_n)$ to accomplish this. The choice of these sequences is based on a thorough study of the neighborhoods in step 1).
3) Since this implies $d_W(h_n, h'_n) \leq 1$ for all $n \in \mathbb{N}$ and $d_V(\phi(h_n), \phi(h'_n)) \xrightarrow{n\to\infty} \infty$, the map ϕ cannot be a quasi-isometry. Hence, the groups H_1 and H_2 are not coorbit equivalent, according to Theorem 6.3.

By following these steps, we get the next corollary, which together with theorem 4.4 implies that distinct members of the two classes of shearlet groups we consider lead to nonequivalent coorbit spaces in almost all cases.

Corollary 6.4 ([14] Corollary 5.5.1, 5.5.2, 5.5.3): If H_1, H_2 are standard or Toeplitz shearlet groups with $H_1 \neq H_2$, then H_1 and H_2 are not coorbit equivalent (in the sense of Definition 6.1).

VII. Conclusion

The isomorphism between decomposition spaces and other function spaces provides a new way to examine their properties as depicted in this note for the class of shearlet coorbit spaces in arbitrary dimension. By using this decomposition space viewpoint it becomes clear that one distinguishing feature of shearlet groups is the way in which the dual action induces a covering of the dual orbit.

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