A Clifford Construction of Multidimensional Prolate Spheroidal Wave Functions

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Abstract—We investigate the construction of multidimensional prolate spheroidal wave functions using techniques from Clifford analysis. The prolates are defined to be eigenfunctions of a certain differential operator and we propose a method for computing these eigenfunctions through expansions in Clifford-Legendre polynomials. It is shown that the differential operator commutes with a time-frequency limiting operator defined relative to balls in n-dimensional Euclidean space.

I. INTRODUCTION

In 1964 [1], the higher dimensional version of the prolates were studied and constructed. After polar coordinates were employed, part of the construction involved determining the eigenvalues of the differential operator \( M_c \) given by

\[
M_c(u)(t) = (1 - t^2)\frac{d^2u}{dt^2} - 2t \frac{du}{dt} + \left( \frac{1}{4} - \frac{N^2}{t^2} \right) - e^2t^2)u = 0,
\]

the solutions of which form the radial part of the higher dimensional prolate. The operator has a singularity at the origin, causing instabilities. Also it is valuable to mention that, in [2], the prolate spheroidal wave functions have been constructed by definition of a new Sturm-Liouville differential operator.

Clifford analysis is a means through which many of the fundamental theorems and techniques of complex analysis can be lifted to higher dimensions (see [3]). In this paper we study the higher-dimensional prolate spheroidal wave functions (PSWFs) through the lens of Clifford analysis.

II. CLIFFORD ANALYSIS

Let \( \{e_1, \ldots, e_m\} \) be the standard basis for \( m \)-dimensional euclidean space \( \mathbb{R}^m \). The non-commutative multiplication in the Clifford algebra \( \mathbb{R}_m \) built over \( \mathbb{R}^m \) is governed by the rules

\[
e_j^2 = -1 \quad j = 1, \ldots, m
\]

\[
e_i e_j = -e_j e_i \quad i \neq j
\]

A canonical base for \( \mathbb{R}_m \) is obtained by considering for any ordered set \( A = \{j_1, j_2, \ldots, j_n\} \subset \{1, \ldots, m\} = M \), the element \( e_A = e_{j_1} e_{j_2} \cdots e_{j_n} \), \( e_\emptyset = 1 \). For example, each \( \lambda \in \mathbb{R}_2 \) may be written as \( \lambda = \lambda_0 + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_1 e_1 e_2 \), where \( \lambda_i \in \mathbb{R} \). The conjugation \( \bar{\lambda} \) of \( \lambda = \sum_{A} \lambda_A e_A \in \mathbb{R}_m \) is given by \( \bar{\lambda} = \sum_{A} \lambda_A e_A \) where \( \bar{e}_j = -e_j \), \( \bar{e}_0 = e_0 \), and \( \alpha \beta = \beta \alpha \) for all \( \alpha, \beta \in \mathbb{R}_m \). The Euclidean space \( \mathbb{R}^m \) is embedded in the Clifford algebra \( \mathbb{R}_m \) by identifying the point \( x = (x_1, \ldots, x_m) \in \mathbb{R}^m \) with the 1-vector \( x = \sum_{j=1}^{m} e_j x_j \).

It should be noted that \( |\lambda_0| \) is the scalar part of the Clifford number \( \lambda \). The product of two 1-vectors splits up into a scalar part and a 2-vector (also called the bivector, part): \( xy = -(x,y) + x \wedge y \) where \( (x,y) = \sum_{j=1}^{m} x_j y_j \) and \( x \wedge y = \sum_{i<j} e_i e_j (x_j y_i - x_i y_j) \).

Note also that if \( x \) is a 1-vector, then \( x^2 = -(x,x) = -|x|^2 \).

Definition 1. Let \( f : \mathbb{R}^m \to \mathbb{R}_m \) be defined and continuously differentiable in an open region \( \Omega \) of \( \mathbb{R}^m \). The Dirac operator \( \partial_x \) is defined on such functions by

\[
\partial_x f = \sum_{j=1}^{m} e_j \partial_x e_j f.
\]

We also allow the Dirac operator to act of the right in the sense that \( f \partial_x \). \( f \) is said to be left (resp. right) monogenic on \( \Omega \) if \( \partial_x f = 0 \) (resp. \( \partial_x f = 0 \)) on \( \Omega \). If \( f \) is left-and right monogenic, we say \( f \) is monogenic. The Dirac operator factorises the Laplace operator in the sense that

\[
\Delta_m = -\partial^2_x.
\]

Definition 2. A left (resp. right) monogenic homogeneous polynomial \( P_k \) of degree \( k \) \((k \geq 0)\) in \( \mathbb{R}^m \) is called a left (resp. right) solid inner spherical monogenic of order \( k \). The set of all left (resp. right) solid inner spherical monogenics of order \( k \) will be denoted by \( M^+_l(k) \), respectively \( M^+_r(k) \).

Lemma 1. Let \( x = \sum_{j=1}^{m} x_j e_j \). For \( P_k \in M^+_l(k) \) and \( s \in \mathbb{N} \) the following fundamental formula holds:

\[\partial_x[x^s P_k] = \begin{cases} -sx^{s-1}P_k & \text{for } s \text{ even} \\ -(s+2k+m-1)x^{s-1}P_k & \text{for } s \text{ odd} \end{cases}\]

For the proof, the reader is referred to [3].

The \( \mathbb{R}_m \)-valued inner product of the functions \( f, g : \mathbb{R}^m \to \mathbb{R}_m \) is given by

\[
\langle f, g \rangle = \int_{\mathbb{R}^m} f(x) g(x) dV(x),
\]

where \( dV \) is the Lebesgue measure on \( \mathbb{R}^m \). The associated norm is given by \( ||f||^2 = \langle f, f \rangle \). The unitary right Clifford-module of Clifford algebra-valued measurable functions on
\( \mathbb{R}^m \) for which \( \|f\|^2 < \infty \) is a right Hilbert Clifford-module which we denote by \( L_2(\mathbb{R}^m, \mathbb{R}_m) \). The multi-dimensional Fourier transform \( \mathcal{F} \) is given by

\[
\mathcal{F}f(\xi) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp(-i(x, \xi)) f(x) dV(x) \tag{3}
\]

for \( f \in L^1(\mathbb{R}^m, \mathbb{R}_m) \) and may be extended unitarily to \( L^2(\mathbb{R}^m, \mathbb{R}_m) \).

**Theorem 2. (Clifford-Stokes theorem)** Let \( f, g \in C_1(\Omega) \). Then for each compact set \( C \subset \Omega \), one has

\[
\int_{\partial C} f(x)n(x)g(x) d\sigma(x) = \int_C ([f(\partial_x)x + f(\partial_x)g]) dV(x)
\]

where \( n(x) \) is the outward pointing unit normal on \( \partial C \) and \( d\sigma \) is the surface area measure on \( \partial C \).

**Proof.** For the proof see [3]. \( \square \)

### III. LEGENDRE POLYNOMIALS

**Definition 3.** Let \( B(1) \) be the closed unit ball in \( \mathbb{R}^m \) and \( \alpha \in \mathbb{R} \) with \( \alpha > -1 \). Then the operator \( D_\alpha \) is defined on continuous functions \( f : B(1) \to \mathbb{R}_m \) by

\[
D_\alpha f(x) = (1 + x^2)^{-\alpha} \partial_x((1 + x^2)^{\alpha+1}f(x)). \tag{4}
\]

**Definition 4.** Let \( n \in \mathbb{N} \), and let \( P_k \in M^+_n(k) \) be fixed. Then we define Clifford-Legendre polynomials \( C_{n,m}^0(P_k)(x) \) as follows:

\[
C_{n,m}^0(P_k)(x) = D_0D_1 \cdots D_{n-1}(P_k)(x). \tag{5}
\]

It is shown in [3] that

\[
C_{n,k}^0(P_k)(x) = C_{n,k}^0(x)P_k(x) \tag{6}
\]

with \( C_{n,k}^0 \in \mathcal{P}_n = \text{span}_{m_n} \{x^s : s \in \mathbb{N}, s \leq n, x \in \mathbb{R}(m)\} \), the space of polynomials having degree less than or equal to \( n \). \( C_{n,k}^0 \) has real coefficients depending on \( k \) and takes value in \( \mathbb{R}_m^0 \oplus \mathbb{R}_m^1 \).

**Theorem 3. (Rodrigues’ Formula)** The Legendre polynomials \( C_{n,k}^0(P_k)(x) \) are also determined by

\[
C_{n,k}^0(P_k)(x) = \partial_x((1 + x^2)^{\alpha}P_k(x)) \tag{7}
\]

**Proof.** For the proof see [3]. \( \square \)

Next we note that the Clifford-Legendre polynomials are eigenfunctions of a second order differential equation.

**Theorem 4.** For all \( n, k \in \mathbb{N} \), there exists a real constant \( C(0, n, k) \) such that

\[
D_0\partial_x(C_n^0(P_k)(x)) = C(0, n, k)C_n^0(P_k)(x),
\]

or equivalently,

\[
\partial_x^2C_n^0(P_k)(x) - 2x\partial_xC_n^0(P_k)(x) - C(0, n, k)C_n^0(P_k)(x) = 0. \tag{8}
\]

**Proof.** For the proof see [3]. \( \square \)

### IV. RESULTS

In this section, we start the process of building multi-dimensional prolate spheroidal wave functions.

**Theorem 5.** The Clifford-Legendre polynomials admit the following explicit representations:

\[
C_{2N}^0(P_k)(x) = \frac{2^{2N}(2N)!}{N!} \sum_{j=0}^{N} \left[ \binom{N}{j} \right] \frac{\Gamma\left(j + \frac{m}{2} + N\right)}{\Gamma\left(j + \frac{m}{2} + 1\right)} (-1)^j |x|^{2j} P_k(x) \tag{9}
\]

and

\[
C_{2N+1}^0(P_k)(x) = -\frac{2^{2N+1}(2N+1)!}{N!} \sum_{j=0}^{N} \left[ \binom{N}{j} \right] \frac{\Gamma\left(j + \frac{m}{2} + N + 1\right)}{\Gamma\left(j + \frac{m}{2} + 1\right)} (-1)^j |x|^{2j} x P_k(x) \tag{10}
\]

**Proof.** By Lemma 1,

\[
\partial_x^2(x^{2j} P_k(x)) = 2j(2j + 2k + m - 2) x^{2j-2} P_k(x),
\]

and

\[
\partial_x^2(x^{2j+1} P_k(x)) = 2j(2j + 2k + m) x^{2j+1} P_k(x).
\]

Therefore, when \( n = 2N + 1 \) is odd, we have

\[
C_{2N+1}^0(P_k)(x) = \partial_x^{2N+1}\left[(1 + x^2)^{2N+1} P_k(x)\right]
\]

\[
= \partial_x^{2N+1}\left[\sum_{j=0}^{2N+1} \binom{2N+1}{j} |x|^{2j} P_k(x)\right]
\]

\[
= \sum_{j=0}^{2N+1} \binom{2N+1}{j} \partial_x^{2N-1}(-2j)(-(2j - 1)
\]

\[
+ 2k + m - 4) x^{2j-2} P_k(x)\]

\[
= \sum_{j=0}^{2N+1} \binom{2N+1}{j} \left[(2j)(2j + 2k + m - 2)(2j - 2)
\]

\[
- \sum_{j=0}^{2N+1} \binom{2N+1}{j} \partial_x^{2N-3} x^{2j-5} x P_k(x)\]

\[
= \sum_{j=N}^{2N+1} \binom{2N+1}{j} \partial_x^{2N-1}((j)(j-1) \cdots (j - (N - 1))
\]

\[
\times (j + k + \frac{m}{2} - 1) \cdots (j + k + \frac{m}{2} - (N - 1))
\]

\[
\times \partial_x^{2j-2N} x P_k(x)\]

\[
= -\sum_{j=0}^{N+1} \binom{2N+1}{j} \partial_x^{2N+1}((j + N)!
\]

\[
\times (j + k + \frac{m}{2} + N - 1)! |x|^{2j-2} x P_k(x)\]

\[
= -\frac{2^{2N+1}(2N+1)!}{N!} \sum_{j=0}^{N} \binom{N}{j} \frac{\Gamma\left(j + \frac{m}{2} + N + 1\right)}{\Gamma\left(j + \frac{m}{2} + 1\right)} (-1)^j |x|^{2j} x P_k(x).
\]
A double application of the Bonnet formula gives
\[ b \text{Bonnet formula for Clifford-Legendre polynomials} \]

\[ \text{Theorem 6. [Bonnet formula for Clifford-Legendre polynomials]} \]

(a) If \( n \) is odd,
\[ xC_{2n+1}^{(0)}(P_k)(x) = \alpha_{n,k} C_{2n+2}^{(0)}(P_k)(x) + \beta_{n,k} C_{2n}^{(0)}(P_k)(x), \]
where \( \alpha_{n,k} = \frac{4^n}{2^{n+k+1}} \), \( \beta_{n,k} = \frac{2(2n+1)(2n+n+k)}{(2n+k+1)} \).

(b) If \( n \) is even,
\[ xC_{2n}^{(0)}(P_k)(x) = \alpha'_{n,k} C_{2n+1}^{(0)}(P_k)(x) + \beta'_{n,k} C_{2n-1}^{(0)}(P_k)(x), \]
where \( \alpha'_{n,k} = \frac{(-2n+n+k)}{2(2n+k+1)(2n+k+2)} \), \( \beta'_{n,k} = \frac{4^n}{2^{n+k+1}} \).

For the proof, the reader is referred to [4].

V. MULTI-DIMENSIONAL PROLATES

Definition 8: Given \( c > 0 \), we define three operators \( L_c \) and \( G_c \) on \( L^2(B(1), \mathbb{R}^m) \) by
\[ G_c f(x) = \chi_B(x) \int_B e^{2\pi i c(x,y)} f(y) dy, \]
and
\[ L_c f(x) = \partial_x((1 - |x|^2)\partial_x f(x)) + 4\pi^2 c^2 |x|^2 f(x). \]

Computation of these eigenvectors, and substitution into (15) gives eigenfunctions of \( L_c \).

The connections between the PSWFs and the eigenfunctions of a multi-dimensional time-frequency limiting operator is now of interest, in particular, the commutation of these operators.

Theorem 7. The operator \( L_c \), defined at (14), is self-adjoint.
Proof. With an application of the Clifford-Stokes formula and the observation that \( (1 - |x|^2) = 0 \) for \( x \in \partial B \), we have
\[ \langle f, L_c g \rangle = \int_B \overline{f(x)} \partial_x((1 - |x|^2)\partial_x g(x)) dx + 4\pi^2 c^2 |x|^2 \overline{f(x)} g(x) dx \]
\[ = - \int_B ((\overline{f(x)} \partial_x) (1 - |x|^2)\partial_x g(x)) dx + \int_B 4\pi^2 c^2 |x|^2 \overline{f(x)} g(x) dx \]
\[ = - \int_{\partial B} ((1 - |x|^2) \partial_x) \overline{f(x)} g(x) d\sigma \]
\[ - \int_{\partial B} ((1 - |x|^2) \overline{f(x)} \partial_x) g(x) d\sigma \]
\[ + \int_B 4\pi^2 c^2 |x|^2 \overline{f(x)} g(x) dx \]
\[ = \int_B \overline{L_c f(x)} g(x) dx = \langle L_c f, g \rangle \]
which completes the proof.

Theorem 8. The operators \( L_c \) and \( G_c \) commute, i.e., \( L_c G_c = G_c L_c \).
Proof. By the definitions of \( L_c \) and \( G_c \) and the self-adjointness...
of \( L_c \), we have

\[
\mathcal{G}_c L_c(f(x)) = \mathcal{G}_c(f(x)) = \chi_B(x) \int_B e^{2\pi ic \langle x, y \rangle} [L_c f(y)] dy
\]

\[
= \chi_B(x) \int_B \left[ (1 - |y|^2) \partial_y (e^{-2\pi ic \langle x, y \rangle}) \right] \partial_y (e^{-2\pi ic \langle x, y \rangle}) dy
\]

\[
+ 4\pi^2 c^2 |y|^2 e^{-2\pi ic \langle x, y \rangle} f(y) dy
\]

\[
= \chi_B(x) \int_B \left[ (1 - |y|^2) \partial_y (e^{-2\pi ic \langle x, y \rangle}) \right] \partial_y (e^{-2\pi ic \langle x, y \rangle}) f(y) dy
\]

\[
+ 4\pi ic y e^{-2\pi ic \langle x, y \rangle} + 4\pi^2 c^2 |y|^2 e^{-2\pi ic \langle x, y \rangle} f(y) dy
\]

\[
= \chi_B(x) \int_B \left[ 4\pi^2 c^2 |x|^2 (e^{2\pi ic \langle x, y \rangle} - \frac{1}{4\pi^2 c^2} \partial_x^2 (e^{2\pi ic \langle x, y \rangle})) - 4\pi ic y e^{2\pi ic \langle x, y \rangle} + \partial_y^2 (e^{2\pi ic \langle x, y \rangle}) \right] f(y) dy
\]

\[
- 2x \partial_x e^{2\pi ic \langle x, y \rangle} - |x|^2 \partial_y^2 (e^{2\pi ic \langle x, y \rangle}) f(y) dy
\]

\[
= \chi_B(x) \int_B \left[ (1 - |x|^2) \partial_x^2 (e^{2\pi ic \langle x, y \rangle}) - 2x \partial_y^2 (e^{2\pi ic \langle x, y \rangle}) + c^2 |x|^2 (e^{2\pi ic \langle x, y \rangle}) \right] f(y) dy = L_c \mathcal{G}_c(f(x)).
\]

\( \square \)

**Theorem 9.** The operators \( \mathcal{G}_c^* \) and \( \mathcal{G}_c^* \mathcal{G}_c \) commute with \( L_c \).

Let \( Q, P_c \) be the orthogonal projections on \( L^2(\mathbb{R}^m, \mathbb{R}_m) \) defined by

\[
Q f(x) = \chi_B(1)(x) f(x);
\]

\[
P_c f(x) = \int_{\mathbb{R}^m} f(y) K_c(x - y) dy
\]

where \( K_c(x) = \int_B e^{2\pi ic \langle x, y \rangle} dy \). The range of \( Q \) is the collection of \( L^2 \) functions that are supported on the ball \( B(1) \) of radius 1 and the range of \( P_c \) is the collection of \( L^2 \) functions whose Fourier transform are supported on the ball \( B(c) \) of radius \( c \). We note that the operator \( \mathcal{G}_c^* \mathcal{G}_c \) is a multiple of the time-frequency operator \( Q P_c \) and conclude from Theorem 9 that \( Q P_c \) commutes with \( L_c \).

**VI. Conclusion and Future Work**

Through consideration of Clifford-Legendre polynomials and proof of an associated Bonnet formula, we developed in this paper sufficient theory to enable the construction of multidimensional Clifford-valued PSWF’s, defined as the eigenfunctions of a self-adjoint differential operator \( L_c \) involving the Dirac operator. We defined time and frequency projections \( Q \) and \( P_c \) and showed that the self-adjoint operator \( Q P_c \) commutes with \( L_c \). It is yet to be shown that the PSWFs are also the eigenfunctions of \( Q P_c \) as there is currently insufficient theory surrounding the linear algebra of Clifford modules to assert that commuting self-adjoint operators share a common eigenbasis – as is the case in Hilbert spaces. Approximation properties of these functions will be explored as has been done in one dimension by Shkolnisky [5] and Xiao, Rokhlin and Yarvin [6].

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