A Clifford Construction of Multidimensional Prolate Spheroidal Wave Functions

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***Abstract*—We investigate the construction of multidimensional prolate spheroidal wave functions using techniques from Clifford analysis. The prolates are defined to be eigenfunctions of a certain differential operator and we propose a method for computing these eigenfunctions through expansions in Clifford-Legendre polynomials. It is shown that the differential operator commutes with a time-frequency limiting operator defined relative to balls in n-dimensional Euclidean space.**

I. Introduction

In 1964 [1], the higher dimensional version of the of prolates were studied and constructed. After polar coordinates were employed, part of the construction involved determining the eigenvalues of the differential operator M_c given by

$$M_c(u)(t) = (1-t^2)\frac{d^2u}{dt} - 2t\frac{du}{dt} + (\frac{\frac{1}{4}-N^2}{t^2} - c^2t^2)u = 0, \tag{1}$$

the solutions of which form the radial part of the higher dimensional prolates. The operator has a singularity at the origin, causing instabilities. Also it is valuable to mention that, in [2], the prolate spheroidal wave functions has been constructed by definition of a new Sturm-Liouville differential operator.

Clifford analysis is a means through which many of the fundamental theorems and techniques of complex analysis can be lifted to higher dimensions (see [3]). In this paper we study the higher-dimensional prolate spheroidal wave functions (PSWFs) thorough the lens of Clifford analysis.

II. Clifford Analysis

Let $\{e_1, \dots, e_m\}$ be the standard basis for m-dimensional euclidean space $\mathbb{R}^m$. The non-commutative multiplication in the Clifford algebra $\mathbb{R}_m$ built over $\mathbb{R}^m$ is governed by the rules

$$\begin{aligned} e_j^2 &= -1 \quad j = 1, \cdots, m \\ e_ie_j &= -e_je_i \quad i \neq j. \end{aligned}$$

A canonical base for $\mathbb{R}_m$ is obtained by considering for any ordered set $A = \{j_1, j_2, \cdots, j_h\} \subset \{1, \cdots, m\} = M$, the element $e_A = e_{j_1}e_{j_2}\cdots e_{j_h}$, $e_\emptyset = 1$. For example, each $\lambda \in \mathbb{R}_2$, may be written as $\lambda = \lambda_0 + \lambda_1e_1 + \lambda_2e_2 + \lambda_{12}e_1e_2$, where $\lambda_i \in \mathbb{R}$. The conjugation $\overline{\lambda}$ of $\lambda = \sum_A \lambda_Ae_A \in \mathbb{R}_m$ is given by $\overline{\lambda} = \sum_A \lambda_A\overline{e_A}$ where $\overline{e_j} = -e_j$, $\bar{e}_\emptyset = e_\emptyset$, and $\overline{\alpha\beta} = \overline{\beta}\,\overline{\alpha}$ for all $\alpha, \beta \in \mathbb{R}_m$. The Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebra $\mathbb{R}_m$ by identifying the point $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$ with the 1-vector $x = \sum_{j=1}^m e_jx_j$. It should be noted that $[\lambda]_0$ is the scalar part of the Clifford number λ. The product of two 1-vectors splits up into a scalar part and a 2-vector (also called the bivector, part): $xy = -\langle x, y\rangle + x \wedge y$ where $\langle x, y\rangle = \sum_{j=1}^m x_jy_j$ and $x \wedge y = \sum_{i<j} e_ie_j(x_iy_j - x_jy_i)$. Note also that if x is a 1-vector, then $x^2 = -\langle x, x\rangle = -|x|^2$.

Definition 1. *Let $f : \mathbb{R}^m \to \mathbb{R}_m$ be defined and continuously differentiable in an open region Ω of $\mathbb{R}^m$. The Dirac operator ∂_x is defined on such functions by*

$$\partial_x f = \sum_{j=1}^m e_j\partial_{x_j}f.$$

We also allow the Dirac operator to act of the right in the sense that $f\partial_x = \sum_{j=1}^m \partial_{x_j}fe_j$. f is said to be left (resp. right) monogenic on Ω if $\partial_x f = 0$ (resp. $f\partial_x = 0$) on Ω. If f is left-and right monogenic, we say f is monogenic. The Dirac operator factorises the Laplace operator in the sense that

$$\Delta_m = -\partial_x^2. \tag{2}$$

Definition 2. *A left (resp. right) monogenic homogeneous polynomial P_k of degree k ($k \geq 0$) in $\mathbb{R}^m$ is called a left (resp. right) solid inner spherical monogenic of order k. The set of all left (resp. right) solid inner spherical monogenics of order k will be denoted by $M_l^+(k)$, respectively $M_r^+(k)$.*

Lemma 1. *Let $x = \sum_{j=0}^m x_je_j$. For $P_k \in M_l^+(k)$ and $s \in \mathbb{N}$ the following fundamental formula holds:*

$$\partial_x[x^sP_k] = \begin{cases} -sx^{s-1}P_k & \text{for } s \text{ even} \\ -(s+2k+m-1)x^{s-1}P_k & \text{for } s \text{ odd.} \end{cases}$$

For the proof, the reader is referred to [3].

The $\mathbb{R}_m$-valued inner product of the functions $f, g : \mathbb{R}^m \to \mathbb{R}_m$ is given by

$$\langle f, g\rangle = \int_{\mathbb{R}^m} \overline{f(x)}g(x)dV(x),$$

where dV is the Lebesgue measure on $\mathbb{R}^m$. The associated norm is given by $\|f\|^2 = [\langle f, f\rangle]_0$. The unitary right Clifford-module of Clifford algebra-valued measurable functions on

$\mathbb{R}^m$ for which $\|f\|^2 < \infty$ is a right Hilbert Clifford-module which we denote by $L_2(\mathbb{R}^m, \mathbb{R}_m)$. The multi-dimensional Fourier transform $\mathcal{F}$ is given by

$$\mathcal{F}f(\xi) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp(-i\langle x, \xi\rangle) f(x) dV(x) \qquad (3)$$

for $f \in L^1(\mathbb{R}^m, \mathbb{R}_m)$ and may be extended unitarily to $L^2(\mathbb{R}^m, \mathbb{R}_m)$.

Theorem 2. ***(Clifford-Stokes theorem)*** *Let $f, g \in C_1(\Omega)$. Then for each compact set $C \subset \Omega$, one has*

$$\int_{\partial C} f(x)n(x)g(x)\, d\sigma(x) = \int_C [(f\partial_x)g + f(\partial_x g)]\, dV(x)$$

where $n(x)$ is the outward pointing unit normal on ∂C and $d\sigma$ is the surface area measure on ∂C.

Proof. For the proof see [3]. □

III. Legendre Polynomials

Definition 3. *Let $B(1)$ be the closed unit ball in $\mathbb{R}^m$ and $\alpha \in \mathbb{R}$ with $\alpha > -1$. Then the operator D_α is defined on continuous functions $f : B(1) \to \mathbb{R}_m$ by*

$$D_\alpha f(x) = (1 + x^2)^{-\alpha} \partial_x((1 + x^2)^{\alpha+1} f(x)). \qquad (4)$$

Definition 4. *Let $n \in \mathbb{N}$, and let $P_k \in M_l^+(k)$ be fixed. Then we define Clifford-Legendre polynomials $C^0_{n,m}(P_k)(x)$ as follows:*

$$C^0_n(P_k)(x) = D_0 D_1 \cdots D_{+n-1}(P_k(x)). \qquad (5)$$

It is shown in [3] that

$$C^0_n(P_k)(x) = C^0_{n,k}(x)P_k(x) \qquad (6)$$

with $C^0_{n,k} \in \mathcal{P}_n = \mathrm{span}_{\mathbb{R}_m}\{x^s : s \in \mathbb{N},\ s \le n,\ x \in \mathbb{R}_{(m)}\}$, the space of polynomials having degree less than or equal to n. $C^0_{n,k}$ has real coefficients depending on k and takes value in $\mathbb{R}^0_m \oplus \mathbb{R}^1_m$.

Theorem 3. *(Rodrigues' Formula) The Legendre polynomials $C^0_n(P_k)(x)$ are also determined by*

$$C^0_n(P_k)(x) = \partial^n_x((1 + x^2)^n P_k(x)) \qquad (7)$$

Proof. For the proof see [3]. □

Next we note that the Clifford-Legendre polynomials are eigenfunctions of a second order differential equation.

Theorem 4. *For all $n, k \in \mathbb{N}$, there exists a real constant $C(0, n, k)$ such that*

$$D_0\partial_x(C^0_n(P_k)(x)) = C(0, n, k)C^0_n(P_k)(x),$$

or equivalently,

$$\partial^2_x C^0_n(P_k)(x) - 2x\partial_x C^0_n(P_k)(x) - C(0, n, k)C^0_n(P_k)(x) = 0. \qquad (8)$$

Proof. For the proof see [3]. □

IV. Results

In this section, we start the process of building multi-dimensional prolate spheroidal wave functions.

Theorem 5. *The Clifford-Legendre polynomials admit the following explicit representations:*

$$C^0_{2N}(P_k)(x) = \frac{2^{2N}(2N)!}{N!} \sum_{j=0}^{N} \left[\binom{N}{j} \times \frac{\Gamma(j + k + \frac{m}{2} + N)}{\Gamma(j + k + \frac{m}{2})} (-1)^j |x|^{2j} P_k(x) \right] \qquad (9)$$

and

$$C^0_{2N+1}(P_k)(x) = -\frac{2^{2N+1}(2N+1)!}{N!} \sum_{j=0}^{N} \left[\binom{N}{j} \frac{\Gamma(j + k + \frac{m}{2} + N + 1)}{\Gamma(j + k + \frac{m}{2} + 1)} (-1)^j |x|^{2j} x P_k(x) \right]. \qquad (10)$$

Proof. By Lemma 1,

$$\partial^2_x(x^{2j}P_k(x)) = 2j(2j + 2k + m - 2)x^{2j-2}P_k(x),$$

and

$$\partial^2_x(x^{2j+1}P_k(x)) = 2j(2j + 2k + m)x^{2j-1}P_k(x).$$

Therefore, when $n = 2N + 1$ is odd, we heve

$$\begin{aligned}
& C^0_{2N+1}(P_k)(x) = \partial^{2N+1}_x[(1 + x^2)^{2N+1}P_k(x)] \\
= \quad & \partial^{2N+1}_x[\sum_{j=0}^{2N+1} \binom{2N+1}{j} |x|^{2j} P_k(x)] \\
= \quad & [\sum_{j=0}^{2N+1} \binom{2N+1}{j} \partial^{2N-1}_x[(-2j)(-(2j-1) \\
+ \quad & 2k + m - 1))x^{2j-2}P_k(x)]] \\
= \quad & [\sum_{j=0}^{2N+1} \binom{2N+1}{j} [(2j)(2j + 2k + m - 2)(2j - 2) \\
& (2j + 2k + m - 4)\partial^{2N-3}_x x^{2j-5} x P_k(x)]] \\
= \quad & [-\sum_{j=N}^{2N+1} \binom{2N+1}{j} 2^{2N-1}[(j)(j-1)\cdots(j-(N-1))] \\
\times \quad & [(j + k + \frac{m}{2} - 1)\cdots(j + k + \frac{m}{2} - (N-1))] \\
\times \quad & \partial^2_x x^{2j-2N} x P_k(x)] \\
= \quad & -\sum_{j=0}^{N+1} \binom{2N+1}{j+N} 2^{2N+1}[\frac{(j+N)!}{(j-1)!} \\
\times \quad & \frac{(j + k + \frac{m}{2} + N - 1)!}{(j + k + \frac{m}{2} - 1)!}] x^{2j-2} x P_k(x) \\
= \quad & -\frac{2^{2N+1}(2N+1)!}{N!} \sum_{j=0}^{N} \binom{N}{j} \frac{\Gamma(j + k + \frac{m}{2} + N + 1)}{\Gamma(j + k + \frac{m}{2} + 1)} \\
& (-1)^j |x|^{2j} x P_k(x).
\end{aligned}$$

The proof is similar when n is even. □

The representation above may be used to provide a Bonnet-type formula for the Clifford-Legendre polynomials, i.e., a formula that expresses $xC_n^{(0)}(x)$ as a linear combination of $C_{n-1}^{(0)}(x)$ and $C_{n+1}^{(0)}(x)$.

Theorem 6. *[Bonnet formula for Clifford-Legendre polynomials]*

(a) *If n is odd,*

$$xC_{2n+1}^0(P_k)(x) = \alpha_{n,k}C_{2n+2}^0(P_k)(x)+\beta_{n,k}C_{2n}^0(P_k)(x), \tag{11}$$

where $\alpha_{n,k} = \frac{-m}{4(\frac{m}{2}+2n+k+1)}$, $\beta_{n,k} = \frac{2(2n+1)(\frac{m}{2}+n+k)}{(\frac{m}{2}+2n+k+1)}$,

(b) *If n is even,*

$$xC_{2n}^0(P_k)(x) = \alpha'_{n,k}C_{2n+1}^0(P_k)(x)+\beta'_{n,k}C_{2n-1}^0(P_k)(x), \tag{12}$$

where $\alpha'_{n,k} = \frac{-(\frac{m}{2}+n+k)}{2(2n+1)(\frac{m}{2}+2n+k)}$, $\beta'_{n,k} = \frac{4n^2}{(\frac{m}{2}+2n+k)}$.

For the proof, the reader is referred to [4].

V. Multi-dimensional prolates

Definition 8: Given $c > 0$, we define three operators L_c and $\mathcal{G}_c$ on $L^2(B(1),\mathbb{R}_m)$ by

$$\mathcal{G}_cf(x) = \chi_B(x)\int_B e^{2\pi ic\langle x,y\rangle}f(y)dy, \tag{13}$$

and

$$L_cf(x) = \partial_x((1-|x|^2)\partial_xf(x)) + 4\pi^2c^2|x|^2f(x). \tag{14}$$

Strictly speaking, L_c is defined on a dense subspace of $L^2(\bar{B}(1),\mathbb{R}_m)$. The m-dimensional Clifford prolates are defined to be the eigenfunctions of L_c. Here we aim to describe an algorithm for their computation and reveal relationships between L_c and certain time-frequency limiting operators defined relative to balls in $\mathbb{R}^m$.

Let $\{Y_{k,j}\}_{j=1}^{d_k}$ be an orthonormal basis for $M_l^+(k)$. Then the Clifford-Legendre polynomials

$$\{R_{n,j,k} = C_n^0(Y_{k,j});\ n\geq 0,\ k\geq 0,\ 0\leq j\leq d_k\}$$

(where d_k is the dimension of $M_l^+(k)$) form an orthonormal basis for $L^2(B(1),\mathbb{R}^m)$. Suppose then that f as an eigenfunction of L_c, i.e., $L_cf = f\chi$ for some $\chi\in\mathbb{R}_m$. We may write

$$f = \sum_{i,k=0}^{\infty}\sum_{j=0}^{d_k}C_i^0(Y_{k,j})b_{i,k,j}, \tag{15}$$

for some constants $b_{i,k,j}\in\mathbb{R}_m$. Then

$$\begin{aligned}L_cf(x) &= \sum_{i,k,j}L_c[C_i^0(Y_{k,j})(x)]b_{i,k,j}\\ &= \sum_{i,k,j}[C(i,k)C_i^0(Y_{k,j})(x) + 4\pi^2c^2x^2C_i^0(Y_{k,j}(x)].\end{aligned} \tag{16}$$

A double application of the Bonnet formula gives

$$\begin{aligned}x^2C_n^0(Y_{k,j}(x)) &= a_{n,k}C_{n+2}^0(Y_{k,j}(x)) + b_{n,k}C_n^0(Y_{k,j}(x))\\ &\quad + c_{n,k}C_{n-2}^0(Y_{k,j}(x)),\end{aligned} \tag{17}$$

for real constants $a_{n,k}$, $b_{n,k}$ and $c_{n,k}$. Substituting (17) into (16) we find that f is an eigenfunction of L_c with eigenvalue χ if and only if $b = (b_{i,k,j})$ is an eigenvector of a multilinear mapping A which is tri-diagonal in the i-variable:

$$Ab = b\chi.$$

Computation of these eigenvectors, and substitution into (15) gives eigenfunctions of L_c.

The connections between the PSWFs and the eigenfunctions of a multi-dimensional time-frequency limiting operator is now of interest, in particular, the commutation of these operators..

Theorem 7. *The operator L_c, defined at* (14)*, is self-adjoint.*

Proof. With an application of the Clifford-Stokes formula and the observation that $(1-|x|^2) = 0$ for $x\in\partial B$, we have

$$\begin{aligned}\langle f, L_cg\rangle &= \int_B\overline{f(x)}[\partial_x((1-|x|^2)\partial_xg(x))\\ &+ 4\pi^2c^2|x|^2g(x)]dx\\ &= -\int_B((\overline{f(x)}\partial_x)(x)(1-|x|^2)\partial_xg(x))dx\\ &+ \int_B\overline{4\pi^2c^2|x|^2f(x)}g(x)dx\\ &= -\{\int_{\partial B}((1-|x|^2)\overline{f(x)}\partial_x)n(x)g(x)d\sigma\\ &- \int_B([(1-|x|^2)(\overline{f(x)}\partial_x)]\partial_x)g(x)dx\}\\ &+ \int_B\overline{4\pi^2c^2|x|^2f(x)}g(x)dx\\ &= \int_B(\overline{\partial_x[(1-|x|^2)(\partial_xf(x))]})g(x)dx\\ &+ \int_B\overline{4\pi^2c^2|x|^2f(x)}g(x)dx\\ &= \int_B\overline{L_cf(x)}g(x)dx = \langle L_cf, g\rangle\end{aligned}$$

which completes the proof. □

Theorem 8. *The operators L_c and $\mathcal{G}_c$ commute, i.e., $L_c\mathcal{G}_c = \mathcal{G}_cL_c$.*

Proof. By the definitions of L_c and $\mathcal{G}_c$ and the self-adjointness of L_c, we have

$$\mathcal{G}_c L_c(f(x)) = \chi_B(x) \int_B e^{2\pi ic<x,y>}[L_c f(y)]dy$$
$$= \chi_B(x) \int_B \overline{[\partial_y((1-|y|^2)\partial_y(e^{-2\pi ic<x,y>}))}$$
$$\overline{+4\pi^2c^2|y|^2e^{-2\pi ic<x,y>}]} f(y)dy$$
$$= \chi_B(x) \int_B \overline{[(1-|y|^2)(4\pi^2c^2|x|^2e^{-2\pi ic<x,y>})}$$
$$\overline{+4\pi icyxe^{-2\pi ic<x,y>} + 4\pi^2c^2|y|^2e^{-2\pi ic<x,y>}]} f(y)dy$$
$$= \chi_B(x) \int_B [4\pi^2c^2|x|^2(e^{2\pi ic<x,y>} - \frac{1}{4\pi^2c^2}\partial_x^2(e^{2\pi ic<x,y>}))$$
$$-4\pi icxye^{2\pi ic<x,y>} + \partial_x^2(e^{2\pi ic<x,y>})]f(y)dy$$
$$= \chi_B(x) \int_B [4\pi^2c^2|x|^2e^{2\pi ic<x,y>} - |x|^2\partial_x^2(e^{2\pi ic<x,y>})$$
$$-2x\partial_x e^{2\pi ic<x,y>} + \partial_x^2(e^{2\pi ic<x,y>})]f(y)dy$$
$$= \chi_B(x) \int_B [(1-|x|^2)\partial_x^2(e^{2\pi ic<x,y>}) - 2x\partial_x(e^{2\pi ic<x,y>})$$
$$+c^2|x|^2(e^{2\pi ic<x,y>})]f(y)dy = L_c\mathcal{G}_c(f(x)).$$

□

Theorem 9. *The operators $\mathcal{G}_c^*$ and $\mathcal{G}_c^*\mathcal{G}_c$ commute with L_c.*

Let Q, P_c be the orthogonal projections on $L^2(\mathbb{R}^m, \mathbb{R}_m)$ defined by

$$Qf(x) = \chi_{B(1)}(x)f(x);$$
$$P_c f(x) = \int_{\mathbb{R}^m} f(y)K_c(x-y)\,dy$$

where $K_c(x) = \int_{B(c)} e^{2\pi ic\langle x,y\rangle}\,dy$. The range of Q is the collection of L^2 functions that are supported on the ball $B(1)$ of radius 1 and the range of P_c is the collection of L^2 functions whose Fourier transform are supported on the ball $B(c)$ of radius c. We note that the operator $\mathcal{G}_c^*\mathcal{G}_c$ is a multiple of the time-frequency operator QP_c and conclude from Theorem 9 that QP_c commutes with L_c.

VI. Conclusion and Future Work

Through consideration of Clifford-Legendre polynomials and proof of an associated Bonnet formula, we developed in this paper sufficient theory to enable the construction of multidimensional Clifford-valued PSWF's, defined as the eigenfunctions of a self-adjoint differential operator L_c involving the Dirac operator. We defined time and frequency projections Q and P_c and showed that the self-adjoint operator QP_c commutes with L_c. It is yet to be shown that the PSWFs are also the eigenfunctions of QP_c as there is currently insufficient theory surrounding the linear algebra of Clifford modules to assert that commuting self-adjoint operators share a common eigenbasis – as is the case in Hilbert spaces. Approximation properties of these functions will be explored as has been done in one dimension by Shkolnisky [5] and Xiao, Rokhlin and Yarvin [6].

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