Adaptive Frames from Quilted Local Time-Frequency Systems

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***Abstract*—A family of regions that cover the time-frequency plane is considered, and from each region, (possibly irregular) sampling points are taken, thereby generating local time-frequency systems for each component region. This results to "local patches" of Gabor systems which are then put together. In this work, we will be looking at different conditions in which the resulting quilted system, as well as its projection onto subspaces of eigenfunctions of time-frequency localization operators, is to exhibit a frame property.**

I. Introduction

Adaptive time-frequency representations [2] have received significant attention in the past several years. With the desire to circumvent some rather stringent properties of traditional time-frequency representations, such as the fixed time-frequency resolution of regular Gabor frames, various constructions have been introduced that allow adaptivity in the time-frequency representation of signals, e.g. [4], [5], [1], [9], to name a few.

In [4], Dörfler introduced the notion of quilted Gabor frames, where a family of Gabor frames and an admissible covering of the time-frequency plane are considered, and on each region of the covering a Gabor frame is assigned, thus a local Gabor system is obtained for each region which are then "quilted" to form a global system. In this way, it would be possible to have different resolutions for different time-frequency components of a signal. Frame conditions for certain cases were then investigated. In this work, we shall present a similar construction that does not assume the pre-existence of Gabor frames, using only assumptions on the density of the sampling points and regularity of the regions that form a covering of the time-frequency plane.

We also show a frame property for a union of possibly irregular local Gabor systems, with each one projected onto a subspace of eigenfunctions of a time-frequency localization operator. These subspaces have optimal time-frequency concentration in the corresponding region on the time-frequency plane. In contrast to similar frames that were studied in [13], the local Gabor systems that we consider need not come from a frame for $L^2(\mathbb{R})$. Finally, we prove a replacement theorem similar to that in [4] but again, there is no assumption that the local system used in the replacement comes from a Gabor frame.

II. Time-Frequency Analysis

We recall in this section some definitions and properties of the short-time Fourier transform, Gabor frames, and time-frequency localization operators. For a detailed discussion on time-frequency analysis, we refer the reader to [8].

A. The Short-Time Fourier Transform and Gabor Frames

The *short-time Fourier transform* (STFT) of $f \in L^2(\mathbb{R}^d)$ with respect to φ is given by

$$\mathcal{V}_\varphi f(z) = \int_{\mathbb{R}^d} f(t)\,\overline{\varphi(t-x)}\,e^{-2\pi i\omega\cdot t}dt = \langle f, \pi(z)\varphi\rangle,$$

where $z = (x,\omega) \in \mathbb{R}^{2d}$ and $\pi(z)$ is the time-frequency shift operator given by $\pi(z)f = f(t-x)\,e^{2\pi i\omega\cdot t}$. The STFT is an isometry from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^{2d})$, i.e. $\|\mathcal{V}_\varphi f\|_2 = \|\varphi\|_2\|f\|_2$, and inversion for the case where $\|\varphi\|_2 = 1$ is given by

$$f = \mathcal{V}_\varphi^* \mathcal{V}_\varphi f = \iint_{\mathbb{R}^{2d}} \mathcal{V}_\varphi f(z)\pi(z)\varphi\, dz, \qquad (1)$$

where the vector-valued integral above and similar expressions in the sequel are understood in a weak sense, cf. [8, Sec. 3.2].

The membership of the STFT in $L^1(\mathbb{R}^{2d})$ provides a definition for the *modulation space* $\mathbf{S}_0(\mathbb{R}^d)$:

$$\mathbf{S}_0(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : \|f\|_{\mathbf{S}_0} := \|\mathcal{V}_{\varphi_0} f\|_1 < \infty\},$$

where $\varphi_0(t) = e^{-\pi\|t\|^2}$. This space is a Banach space continuously embedded on $L^2(\mathbb{R}^d)$ and $L^1(\mathbb{R}^d)$, and is isometrically invariant under time-frequency shifts and the Fourier transform, cf. [7]. Some conditions for membership of f in $\mathbf{S}_0(\mathbb{R}^d)$ include f being bandlimited and belonging to $L^1(\mathbb{R}^d)$, or both fw_s and $\hat{f}w_s$ belonging to $L^2(\mathbb{R}^d)$, where $\hat{f}$ is the Fourier transform of f and $w_s(t) = (1+t^2)^{s/2}$, $s > d$.

A sequence $\{e_j\}_{j\in J}$ in a separable Hilbert space $\mathcal{H}$ is a *frame* if there exist positive constants $A, B > 0$, called *lower* and *upper frame bounds*, respectively, such that for all $f \in \mathcal{H}$

$$A\|f\|_2^2 \le \sum_{j\in J} |\langle f, e_j\rangle|^2 \le B\|f\|_2^2.$$

Given a sequence (not necessarily a frame) $\{e_j\}_{j\in\mathcal{J}}$ in $\mathcal{H}$, the *analysis, and frame operators* C and S are given by $C : f \mapsto \{\langle f, e_j\rangle\}_{j\in\mathcal{J}}$ and $S : f \mapsto \sum_{j\in\mathcal{J}}\langle f, e_j\rangle e_j$ respectively. The adjoint C^* is called the *synthesis operator*, and it can be shown that $C^* : \{c_j\}_{j\in\mathcal{J}} \mapsto \sum_{j\in\mathcal{J}} c_j e_j$ and $S = C^*C$.

Furthermore, if the sequence is a frame, then the associated frame operator S is invertible, and $\{S^{-1}e_j\}_{j\in\mathcal{J}}$ is also a frame, called the *canonical dual frame*. Moreover for every $f \in \mathcal{H}$, perfect reconstruction is guaranteed by the following unconditionally convergent series: $f = \sum_{j\in\mathcal{J}}\langle f, S^{-1}e_j\rangle e_j$, and $f = \sum_{j\in\mathcal{J}}\langle f, e_j\rangle S^{-1}e_j$.

Given a window function $\varphi \in L^2(\mathbb{R}^d)$ and a countable point set $\Gamma \in \mathbb{R}^{2d}$, the *Gabor system* $\mathcal{G}(g,\Gamma)$ is given by $\mathcal{G}(g,\Gamma) = \{\pi(\lambda)g : \lambda \in \Gamma\}$. In contrast to the case where Γ is a lattice, for which $\mathcal{G}(g,\Gamma)$ is called *regular Gabor system*, the point set Γ that we consider need not have any special structure, and the resulting system is called an *irregular Gabor system*. We say that $\mathcal{G}(\varphi,\Gamma)$ is a *Gabor frame* if $\mathcal{G}(\varphi,\Gamma)$ is a frame.

B. Time-Frequency Localization

In this section, we review some results on time-frequency localization operators cf. [3], [10].

Let Ω be a compact set in $\mathbb{R}^{2d}$, χ_Ω the characteristic function on Ω, and φ a window function in $L^2(\mathbb{R}^d)$, with $\|\varphi\|_2 = 1$. The *time-frequency localization operator* $H_{\Omega,\varphi}$ is defined by

$$H_{\Omega,\varphi}f = \iint_\Omega \mathcal{V}_\varphi f(z)\pi(z)\varphi\, dz = \mathcal{V}_\varphi^*\left(\chi_\Omega\, \mathcal{V}_\varphi f\right).$$

The above integral can be interpreted as the portion of the function f that is essentially contained in Ω. Given $\varepsilon > 0$, a function f is said to be (ε,φ)-*concentrated* on Ω if

$$\langle H_{\Omega,\varphi}f, f\rangle = \iint_\Omega |\mathcal{V}_\varphi f(z)|^2 dz \geq (1-\varepsilon)\|f\|_2^2. \tag{2}$$

The time-frequency localization operator H is a compact and self-adjoint operator so we can consider the spectral decomposition $H_{\Omega,\varphi}f = \sum_{k\in\mathbb{N}}\alpha_k\langle f,\psi_k\rangle\psi_k$, where the eigenvalues $\alpha_k \in [0,1]$, $k \in \mathbb{N}$ are arranged in a decreasing order and $\{\psi_k\}_{k=1}^\infty$ are the corresponding orthonormal eigenfunctions. By the min-max theorem for compact, self-adjoint operators, the first eigenfunction has optimal time-frequency concentration inside Ω in the sense of (2), i.e.

$$\iint_\Omega |\mathcal{V}_\varphi\psi_1(z)|^2 dz = \max_{\|f\|_2=1}\iint_\Omega |\mathcal{V}_\varphi f(z)|^2 dz.$$

If we let V_N be the span of the first N eigenfunctions and if $f \in V_N$, so $f = \sum_{k=1}^N\langle f,\psi_k\rangle\psi_k$, then

$$\langle H_{\Omega,\varphi}f, f\rangle = \sum_{k=1}^N \alpha_k|\langle f,\psi_k\rangle|^2 \geq \alpha_N\|f\|_2^2,$$

so that elements in V_N are $(1-\alpha_N,\varphi)$-concentrated on Ω.

III. Frames from Quilted Local Gabor Systems

In this section, we present our construction of adaptive frames from local, possibly irregular, Gabor systems. Let $\lambda \in \Lambda$ be a countable index set, and consider the family $\{\Omega_\lambda\}_{\lambda\in\Lambda}$ of compact subsets of the time-frequency plane $\mathbb{R}^{2d}$ that satisfy the following:

1) $\bigcup_{\lambda\in\Lambda}\Omega_\lambda = \mathbb{R}^{2d}$,
2) $\sup_{\lambda\in\Lambda}\#\{r \in \Lambda : \Omega_r \cap \Omega_\lambda \neq \emptyset\} < \infty$.

Any family of sets satisfying the above conditions are called *admissible covers.* Our view is to sample finitely many points, say $\mathcal{F}_\lambda \subset \Omega_\lambda$, on each of the compact sets Ω_λ with different associated window function $g_\lambda \in L^2(\mathbb{R}^d)$, and do analysis on the collated points to obtain the local time-frequency system $\{\pi(p_\lambda)g_\lambda\}_{\lambda\in\mathcal{F}_\lambda}$. Synthesis is then done by doing 'local syntheses' on each of the region, and then summing all of it up on Λ. That is, we want to 'quilt' such systems to generate a frame for $L^2(\mathbb{R}^d)$, hence we want constants $A, B > 0$ such that for any $f \in L^2(\mathbb{R}^d)$.

$$A\|f\|_2^2 \leq \sum_{\lambda\in\Lambda}\sum_{p_\lambda\in\mathcal{F}_\lambda}|\langle f,\pi(p_\lambda)g_\lambda\rangle|^2 \leq B\|f\|_2^2. \tag{3}$$

Indeed, this construction is quite similar to quilted Gabor frames [4], or the more general system of Gabor molecules in [11], however we do not assume that the local time-frequency systems come from pre-existing Gabor frames.

By adding some moderate hypotheses on the density of the sample set $\bigcup_{\lambda\in\Lambda}\mathcal{F}_\lambda$ and a uniform decay condition on each g_λ, one can construct the frame bounds on (3). In particular, let $K > 0$ and $s > 2d$, define

$$D_{s,K} := \{h \in L^2(\mathbb{R}^d) : |V_{\varphi_0}h(z)| \leq K(1+\|z\|^2)^{-s/2}\}, \tag{4}$$

where $\varphi_0(t) = e^{-\pi\|t\|^2}$.

The following result, based on [4, Theorem 1], although originally proven under the assumption of pre-existing Gabor frames for each local systems, still holds true even when they are omitted. One needs however, that the sample set $\bigcup_{\lambda\in\Lambda}\mathcal{F}_\lambda$ form a *relatively separated* set, that is, one must be able to write it as a finite union of sets $\{S_k \subset \mathbb{R}^{2d} : k = 1,2,...,K\}$ such that $\inf\{|x-y| : x,y \in S_k\} > 0$ for each k. Now, it can be shown that $\sup_{\lambda\in\Lambda}\#\mathcal{F}_\lambda < \infty$ implies that $\bigcup_{\lambda\in\Lambda}\mathcal{F}_\lambda$ is relatively separated.

Theorem III.1. *An upper frame bound satisfying B in* (3) *exists, provided* $\sup_{\lambda\in\Lambda}\#\mathcal{F}_\lambda < \infty$ *and* $\{g_\lambda\}_{\lambda\in\Lambda} \subseteq D_{s,K}$.

Next we give necessary definitions and show some results regarding the density of the set of sampling points in anticipation for a proof of the existence of the lower frame bound. In what follows, we assume the same uniform decay $\{g_\lambda\}_{\lambda\in\Lambda} \subseteq D_{s,K}$. The results are based on techniques by Feichtinger and Sun [6]. We define $E \subseteq \mathbb{R}^{2d}$ to be a rectangle if it is of the form $E = \prod_{k=1}^d\mathcal{I}_k$ where each $\mathcal{I}_k$ is an interval in $\mathbb{R}$. Each $\mathcal{I}_k$ of a rectangle E is also called a *side* of E. For a $\delta > 0$, we also say that a countable subset $\Lambda \subset \mathbb{R}^d$ is a δ-dense set if $\bigcup_{\lambda\in\Lambda}\prod_{k=1}^\infty\left[\lambda_k - \frac{\delta}{2}, \lambda_k + \frac{\delta}{2}\right] = \mathbb{R}^d$.

The following lemma is a generalization of Lemma 3.5 in [6] to a system with varying windows satisfying (4). The proof is similar.

Lemma III.2. *Let* $\{E_n\}_{n\in\mathbb{Z}^{2d}}$ *be a family of rectangles in* $\mathbb{R}^{2d}$ *such that* $\bigcup_{n\in\mathbb{Z}^{2d}} E_n = \mathbb{R}^{2d}$, *and* $|E_n \cap E_m| = 0$ *whenever* $n \neq m$. *Suppose further that the side lengths of each* E_n *are* *no greater than some $\delta > 0$. For each $n \in \mathbb{Z}^{2d}$, let $p_n \in E_n$, and $g_n \in L^2(\mathbb{R}^d)$ such that $\{g_n\}_{n\in\mathbb{Z}^{2d}} \subseteq D_{s,K}$. If $0 < \inf_{n\in\mathbb{Z}^{2d}} \|g_n\|_2 \leq \sup_{n\in\mathbb{Z}^{2d}} \|g_n\|_2 < \infty$, and δ is small enough, then $\{|E_n|^{1/2}\pi(p_n)g_n : n \in \mathbb{Z}^{2d}\}$ is a frame for $L^2(\mathbb{R}^d)$.*

For the main result of this section, we add another assumption on the behavior of the admissible covering $\{\Omega_\lambda\}_{\lambda\in\Lambda}$, namely, $\sup_{\lambda\in\Lambda} \operatorname{Diam}\Omega_\lambda < \infty$.

Theorem III.3. *For each $\lambda \in \Lambda$, let $\mathcal{E}_\lambda$ be a finite family of rectangles that cover Ω_λ such that $|E_\lambda \cap E'_\lambda| = 0$ whenever $E_\lambda \neq E'_\lambda \in \mathcal{E}_\lambda$. Furthermore, let $\mathcal{F}_\lambda$ be a set of finite points inside Ω_λ satisfying the following properties: $\sup_{\lambda\in\Lambda} \#\mathcal{F}_\lambda < \infty$ and that for any $E_\lambda \in \mathcal{E}_\lambda$, there exists $p_\lambda \in \mathcal{F}_\lambda$ such that $p_\lambda \in E_\lambda \cap \Omega_\lambda$. If $\sup_{\lambda\in\Lambda} \operatorname{Diam}\Omega_\lambda < \infty$, then the sample set $\bigcup_{\lambda\in\Lambda} \mathcal{F}_\lambda$ is a δ-dense set for $\mathbb{R}^{2d}$ for some $\delta > 0$.*

Consequently, given a family of windows $\{g_\lambda\}_{\lambda\in\Lambda} \subseteq D_{s,K}$ with $\|g_\lambda\|_2 = 1$ for all $\lambda \in \Lambda$, then for a sufficiently small δ, the irregular Gabor system $\bigcup_{\lambda\in\Lambda}\{\pi(p_\lambda)g_\lambda : p_\lambda \in \mathcal{F}_\lambda\}$ is a frame for $L^2(\mathbb{R}^d)$.

Proof. Let $D := \sup_{\lambda\in\Lambda} \operatorname{Diam}\Omega_\lambda < \infty$. Define

$$r := \sup_{\lambda\in\Lambda} \max_{E_\lambda\in\mathcal{E}_\lambda} \operatorname{Diam}(E_\lambda \cap \Omega_\lambda) \leq D < \infty.$$

We now show that $\bigcup_{\lambda\in\Lambda} \mathcal{F}_\lambda$ is $2r$-dense. That is, $\mathbb{R}^{2d} = \bigcup_{\lambda\in\Lambda} \bigcup_{p_\lambda\in\mathcal{F}_\lambda} \prod_{i=1}^{2d} [(p_\lambda)_i - r, (p_\lambda)_i + r]$. Hence we let $z \in \mathbb{R}^{2d}$, then $z \in \Omega_\lambda$ for some $\lambda \in \Lambda$, and furthermore $z \in E_\lambda \cap \Omega_\lambda$ for some $E_\lambda \in \mathcal{E}_\lambda$ by construction. Therefore, for any $p_\lambda \in \mathcal{F}_\lambda$,

$$|z - p_\lambda| \leq \max_{\{E_\lambda\in\mathcal{E}_\lambda\}} \operatorname{Diam}(E_\lambda \cap \Omega_\lambda) \leq r < \infty,$$

which implies that for all $i \in \{1, ..., 2d\}$, $|z_i - (p_\lambda)_i| \leq r$. Therefore $z \in \prod_{i=1}^{2d} [(p_\lambda)_i - r, (p_\lambda)_i + r]$. Hence we can take $\delta := 2r$.

Next, let $E_n = n\delta + \delta[0,1]^{2d}$, where $n \in \mathbb{Z}^{2d}$. Thus $\{E_n\}_{n\in\mathbb{Z}^{2d}}$ covers $\mathbb{R}^{2d}$ by rectangles of side lengths δ, and $|E_n \cap E_m| = 0$ whenever $n \neq m$. Since $\mathbb{R}^{2d} = \bigcup_{\lambda\in\Lambda} \bigcup_{p_\lambda\in\mathcal{F}_\lambda} \prod_{i=1}^{2d} [(p_\lambda)_i - \frac{\delta}{2}, (p_\lambda)_i + \frac{\delta}{2}]$, then any rectangle of side length δ will intersect at least one point in $\bigcup_{\lambda\in\Lambda} \mathcal{F}_\lambda$. Hence for each $n \in \mathbb{Z}^{2d}$, E_n will intersect with $\mathcal{F}_{\lambda'}$ for some $\lambda' \in \Lambda$, in which case, we set $p_n = p_{\lambda'}$ and $g_n = g_{\lambda'}$. Therefore by Lemma III.2, for a small enough $\delta > 0$, we have a lower frame bound $A' > 0$ such that for all $f \in L^2(\mathbb{R}^{2d})$ $A'||f||_2^2 \leq \sum_{n\in\mathbb{Z}^{2d}} |\langle f, |E_n|^{1/2}\pi(p_n)g_n\rangle|^2$. Then by Theorem III.1 and $\sup_{n\in\mathbb{Z}^{2d}} |E_n| \leq \delta^{2d}$, we have some $B > 0$ and $A := \frac{A'}{\delta^{2d}} > 0$ such that $A||f||_2^2 \leq \sum_{n\in\mathbb{Z}^{2d}} |\langle f, \pi(p_n)g_n\rangle|^2 \leq \sum_{\lambda\in\Lambda} \sum_{p_\lambda\in\mathcal{F}_\lambda} |\langle f, \pi(p_\lambda)g_\lambda\rangle|^2 \leq B||f||_2^2$, hence $\bigcup_{\lambda\in\Lambda}\{\pi(p_\lambda)g_\lambda : p_\lambda \in \mathcal{F}_\lambda\}$ is a frame for $L^2(\mathbb{R}^d)$. ☐

This shows that for well-behaving regions that cover the time-frequency plane, one can construct adaptive frames by taking points inside sufficiently dense 'grids' $\{E_\lambda \cap \Omega_\lambda : E_\lambda \in \mathcal{E}_\lambda\}$ in each of the local regions and then 'quilting' all the generated local time-frequency systems.

IV. Frames from Quilted TF-localized Systems

In this section we consider quiltings that involve local Gabor systems projected onto optimally concentrated subspace of a time-frequency localization operator in each region of interest. We begin with the following lemma.

Lemma IV.1 ([12, Lemma 4.4])**.** *Consider the time-frequency localization operator $H_{\Omega,\varphi}$ with eigenvalue-eigenfunction pairs $\{(\alpha_k, \psi_k)\}_{k=1}^\infty$ where $\varphi \in \mathbf{S}_0(\mathbb{R})$, $\Omega \subset \mathbb{R}^2$ such that $\|\varphi\|_2 = 1$ and $|\Omega| > 1$. Let $N \in \mathbb{N}$, $\mathcal{F}_\Omega \subset \Omega$ be a set of finite points in Ω, and $g \in L^2(\mathbb{R}^d)$ be another window function such that $\|g\|_2 = 1$. If there exists some $\nu \in (0, \alpha_N)$ such that for all $p \in V_N = \operatorname{span}\{\psi_1, ..., \psi_N\}$*

$$\frac{1}{\#\mathcal{F}_\Omega} \sum_{\lambda\in\mathcal{F}_\Omega} |\langle p, \pi(\lambda)g\rangle|^2 \geq \frac{\langle H_{\Omega,\varphi}p, p\rangle - \nu\|p\|_2^2}{|\Omega|}, \quad (5)$$

then there exist $A, B > 0$ such that for all $f \in L^2(\mathbb{R})$,

$$A\|P_{V_N}f\|_2^2 \leq \sum_{\lambda\in\mathcal{F}_\Omega} |\langle f, P_{V_N}\pi(\lambda)g\rangle|^2 \leq B\|P_{V_N}f\|_2^2. \quad (6)$$

Remark.

1) It was shown in [12] that with a high and controllable probability, a sufficiently dense set of local random samples $\mathcal{F}_\Omega$ inside Ω satisfies inequality (5).
2) If (6) holds, then $\{P_{V_N}\pi(\lambda)g\}_{\lambda\in\mathcal{F}_\Omega}$ is a frame for V_N. This fact will be used for our subsequent results.
3) It can be shown that in (6), $A = \frac{\#\mathcal{F}_\Omega(\alpha_N - \nu)}{|\Omega|}$ and $B = \#\mathcal{F}_\Omega$ are sufficient frame bounds. It can also be assumed without loss of generality that $\nu \in (0, \alpha_N)$ satisfies $\alpha_N - \nu \leq |\Omega|$.

An immediate application of Lemma IV.1 is when one considers a family of time frequency localization operators $\{H_{\Omega_\mu,\varphi}\}_{\mu\in\Lambda}$ where Λ is a countable index set, and the family $\{\Omega_\mu\}_{\mu\in\Lambda}$ forms an admissible cover for $\mathbb{R}^2$, with $\varphi \in \mathbf{S}_0(\mathbb{R})$ and $\|\varphi\|_2 = 1$. It has been shown in [5] that under some regularity conditions on the regions Ω_μ, and the localization operators $H_{\Omega_\mu,\varphi}$: for each $\mu \in \Lambda$, there exists $N_\mu \in \mathbb{N}$ and constants $C, D > 0$ such that

$$C\|f\|_2^2 \leq \sum_{\mu\in\Lambda} \sum_{k=1}^{N_\mu} |\langle f, \phi_k^{(\mu)}\rangle|^2 \leq D\|f\|_2^2 \quad (7)$$

where for each $H_{\Omega_\mu,\varphi}$, its eigenvalue-eigenvector pairs are given by $\left\{\left(\alpha_k^{(\mu)}, \psi_k^{(\mu)}\right)\right\}_{k=1}^\infty$. The following theorem is a direct application of inequality (6) along with the norm equivalence (7).

Theorem IV.2. *Suppose $2 < \inf_{\mu\in\Lambda} |\Omega_\mu| \leq \sup_{\mu\in\Lambda} |\Omega_\mu|$. If for each $\mu \in \Lambda$, $H_{\Omega_\mu,\varphi}$ along with $N_\mu \in \mathbb{N}, \nu_\mu \in (0, \alpha_{N_\mu}^{(\mu)}), g_\mu \in L^2(\mathbb{R})$, and finite points $\mathcal{F}_\mu \subset \Omega_\mu$ satisfy a sampling inequality (5) with $0 < \inf_{\mu\in\Lambda}\{\alpha_{N_\mu}^{(\mu)} - \nu_\mu\}$ and $0 < \inf_{\mu\in\Lambda} \#\mathcal{F}_\mu \leq \sup_{\mu\in\Lambda} \#\mathcal{F}_\mu < \infty$, then the system $\bigcup_{\mu\in\Lambda}\{P_{V_{N_\mu}}\pi(\lambda)g_\mu : \lambda \in \mathcal{F}_\mu\}$ is a frame for $L^2(\mathbb{R})$.*

Proof. Choose $A = \frac{\inf_{\mu\in\Lambda}\#\mathcal{F}_\mu\cdot\inf_{\mu\in\Lambda}\{\alpha^{(\mu)}_{N_\mu}-\nu_\mu\}}{\sup_{\mu\in\Lambda}|\Omega_\mu|}$ and $B = \sup_{\mu\in\Lambda}\#\mathcal{F}_\mu$. It follows from Remarks 2 and 3 above that

$$A\sum_{k=1}^{N_\mu}|\langle f,\psi_k^{(\mu)}\rangle|^2 \le \sum_{\lambda\in\mathcal{F}_\mu}|\langle f, P_{V_{N_\mu}}\pi(\lambda)g_\mu\rangle|^2 \le B\sum_{k=1}^{N_\mu}|\langle f,\psi_k^{(\mu)}\rangle|^2.$$

By (7), and summing the above inequality in μ, the theorem follows. □

Now, let us consider the modified replacement theorem [4, Theorem 3], whose proof, with proper modifications, holds almost verbatim.

Theorem IV.3. *Let $\mathcal{G}_1 = \{e_i : i \in \mathcal{I}\}$ be a frame on $L^2(\mathbb{R}^d)$, $\mathcal{F}_1 \subseteq \mathcal{I}$ be a finite subset of $\mathcal{I}$, and $\mathcal{G}_2 = \{h_j : j \in \mathcal{F}_2\}$ be a finite subset of $L^2(\mathbb{R}^d)$.*

Suppose A is the lower frame bound for $\mathcal{G}_1$, $C^*_{\mathcal{F}_1} : \{c_i\}_{i\in\mathcal{I}} \in \ell^2(\mathcal{F}_1) \mapsto \sum_{\mathcal{F}_1} c_ie_i \in L^2(\mathbb{R})$, and $C^*_{\mathcal{F}_2} : \{c_j\}_{j\in\mathcal{F}_2} \in \ell^2(\mathcal{F}_2) \mapsto \sum_{\mathcal{F}_2} c_jh_j \in L^2(\mathbb{R})$ are the corresponding synthesis operators for $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. If there exists a linear mapping $L : \ell^2(\mathcal{F}_1) \to \ell^2(\mathcal{F}_2)$ in which

$$\|C^*_{\mathcal{F}_1} - C^*_{\mathcal{F}_2}L\|_{\ell^2(\mathcal{F}_1)\to L^2(\mathbb{R})} < \frac{A}{2}, \tag{8}$$

then $\{e_i : \mathcal{I}\setminus\mathcal{F}_1\} \cup \{h_j : j \in \mathcal{F}_2\}$ is a frame for $L^2(\mathbb{R})$.

The next lemma shows that given a region $\Omega \subset \mathbb{R}^2$, it is possible to find a larger region $\Omega^* \supseteq \Omega$ in which a local linear sum of time-frequency shifts of a window g by points on Ω is concentrated on Ω^*.

Lemma IV.4. *Let $g,\varphi \in D_{s,K}$ with $\|g\|_2 = \|\varphi\|_2 = 1$. Let Ω be a compact set in $\mathbb{R}^2$ and $\mathcal{F}_1$ a finite set of points in Ω. Then given $\varepsilon > 0$, there exists $\Omega^* \supseteq \Omega$ and $N \in \mathbb{N}$ such that $\|f - P_{V_N}f\|_2 < \varepsilon$ for any $f \in \{\sum_{\lambda\in\mathcal{F}_1} a_\lambda\pi(\lambda)g : \sum_{\lambda\in\mathcal{F}_1}|a_\lambda|^2 = 1\}$, where V_N is the subspace of the first N eigenfunctions of $H_{\Omega^*,\varphi}$.*

We will show that some local elements in a Gabor frame in a region can be replaced by another time-frequency system, subject to some concentration requirement.

Theorem IV.5. *Let $g \in D_{s,K}$ with $\|g\|_2 = 1$, Γ a countable set of points in $\mathbb{R}^2$, such that $\mathcal{G}(g,\Gamma)$ is a Gabor frame with lower frame bound A. Let $\Omega \subset \mathbb{R}^2$ be compact with finite points $\mathcal{F}_1 = \Omega\cap\Gamma$. Suppose $\varphi \in D_{s,K}$, $\Omega^* \supseteq \Omega$, and $N \in \mathbb{N}$ are chosen so that $H_{\Omega^*,\varphi}$ along with the subspace V_N satisfy the conclusion of Lemma IV.4 in which $\varepsilon < \frac{A}{2}$. Furthermore if $H_{\Omega^*,\varphi}$ and N along with $h \in L^2(\mathbb{R})$, and finite points $\mathcal{F}_2 \subset \Omega^*$ also satisfy the sampling inequality* (5) *then*

$$\{\pi(\lambda)g : \lambda \in \Gamma\setminus\mathcal{F}_1\} \cup \{P_{V_N}\pi(\mu)h : \mu \in \mathcal{F}_2\}$$

is a frame for $L^2(\mathbb{R})$.

Proof. We use Theorem IV.3. Let $C^*_{\mathcal{F}_1} : \{c_\lambda\}_{\lambda\in\mathcal{F}_1} \mapsto \sum_{\lambda\in\mathcal{F}_1} c_\lambda\pi(\lambda)g$ and $C^*_{\mathcal{F}_2} : \{c_\mu\}_{\mu\in\mathcal{F}_2} \mapsto \sum_{\mu\in\mathcal{F}_2} c_\mu P_{V_N}\pi(\mu)h$ be the corresponding synthesis operators. By hypothesis and Lemma IV.1, $\{P_{V_N}\pi(\mu)h : \mu \in \mathcal{F}_2\}$ is a frame on V_N, and therefore the associated frame operator S_{V_N} on V_N is invertible with inverse T_{V_N} on V_N. Moreover, we define $L : \ell^2(\mathcal{F}_1) \to \ell^2(\mathcal{F}_2)$ to be the mapping

$$L : c \mapsto \left\{\left\langle \sum_{\lambda\in\mathcal{F}_1} P_{V_N}c_\lambda\pi(\lambda)g, T_{V_N}P_{V_N}\pi(\mu)h\right\rangle\right\}_{\mu\in\mathcal{F}_2}. \tag{9}$$

The linearity of L easily follows from the linearity of P_{V_N}. Proving boundedness is straightforward, with $\|L\|_{\ell^2(\mathcal{F}_1)\to\ell^2(\mathcal{F}_2)} \le \sqrt{\#\mathcal{F}_1\cdot\#\mathcal{F}_2}\|T_{V_N}\|_{V_N\to V_N}$. Let $a = \{a_\lambda\}_{\lambda\in\mathcal{F}_1}$ with $\|a\|_{\ell^2} = 1$. Then

$$\|(C^*_{\mathcal{F}_1} - C^*_{\mathcal{F}_2}L)a\|_2 = \left\|\sum_{\lambda\in\mathcal{F}_1} a_\lambda\pi(\lambda)g - P_{V_N}\sum_{\lambda\in\mathcal{F}_1} a_\lambda\pi(\lambda)g\right\|_2 < \varepsilon.$$

Therefore $\|(C^*_{\mathcal{F}_1} - C^*_{\mathcal{F}_2}L)\|_{\ell^2(\mathcal{F}_2)\to L^2(\mathbb{R})} \le \varepsilon < \frac{A}{2}$ as required. □

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