Signal transmission through an unidentified channel

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***Abstract*—We formulate and study the problem of recovering a signal x in $\mathcal{X} \subset \mathbb{C}^L$ which, after adding with a pilot signal $c \in \mathbb{C}^L \backslash \{0\}$, is transmitted through an unknown channel H in $\mathcal{H} \subset \mathcal{L}(\mathbb{C}^L, \mathbb{C}^L)$. Here, $\mathcal{X}$ and $\mathcal{H}$ are a priori known and fixed while c is designed by the user. In particular, we consider the case where $\mathcal{H}$ is generated by a subset of time-frequency shift operators on $\mathbb{C}^L$, which leads to investigation of properties of Gabor matrices.**

I. Introduction

To transmit data efficiently over frequency-selective and time-varying channels, a communication channel is generally identified or estimated before its use. For this, communication systems often adapt the following two stage transmission scheme. In the first stage, a known pilot signal is transmitted based on which the channel is identified or estimated [2], [5], [11]. In the second stage, the actual data signal is transmitted through the channel and the receiver uses the channel information to recover the data signal. However, for rapidly varying channels such a scheme is no longer applicable. In that case it may be of advantage to combine the two stages so as to estimate the channel and the data signal simultaneously.

This paper investigates one of such signal recovery schemes. In our model, channels and data signals are assumed to be in some known sets, and the data signal is combined additively with a pilot signal before transmitting it through the channel. As we shall see, the design freedom in the pilot signal is what enables the exact recovery of the data signal even if the data signal cannot be recovered from its corresponding output (see Example 10 below).

While deriving some necessary and/or sufficient conditions for the recovery of data signals that passed through an unidentified channel in the general setup, we consider the application relevant case where the channel space is spanned by a subset of time-frequency shift operators on $\mathbb{C}^L$. This leads to the investigation of properties of Gabor matrices [10]. The special case where the channel space is spanned only by time shift operators on $\mathbb{C}^L$ corresponds to circular convolutions; the study of simultaneous channel identification and signal recovery in this case is referred to as blind deconvolution [1], [4], [8], [13]. We also give some examples to illustrate our results.

II. Background and Problem Formulation

A. Channel Identification and Signal Recovery

In communications engineering, it is often required to identify a channel before using it to transmit signals.

Definition 1. *A class of linear operators $\mathcal{H} \subset \mathcal{L}(\mathbb{C}^L, \mathbb{C}^L)$ is said to be* identifiable *if there exists a vector $c \in \mathbb{C}^L$ such that the map*

$$\Phi_c : \mathcal{H} \longrightarrow \mathbb{C}^L, \quad H \mapsto Hc$$

is injective. Such a vector c is called an identifier *for $\mathcal{H}$.*

Once the communication channel is identified, we use it to transmit a signal x from a set $\mathcal{X} \subset \mathbb{C}^L$. The receiver observes the channel output $y = Hx$ where the information of H is now known, and therefore x can be successfully recovered from y provided that H is injective on $\mathcal{X}$.

B. Problem Formulation

We aim to recover signals that are transmitted through an unidentified channel. To achieve this, we combine the process of channel identification and signal recovery by modeling the input signal to be of the form $x + c$, where $x \in \mathcal{X}$ is the data signal to be sent and $c \in \mathbb{C}^L \backslash \{0\}$ is a pilot signal which is designed by the user. We formulate our problem precisely as follows.

Main Problem. What conditions on $\mathcal{H} \subset \mathcal{L}(\mathbb{C}^L, \mathbb{C}^L)$ and $\mathcal{X} \subset \mathbb{C}^L$ are necessary and/or sufficient so that there exists a vector $c \in \mathbb{C}^L \backslash \{0\}$ with the property that $x \in \mathcal{X}$ can be recovered uniquely from $y = H(x+c)$ with $H \in \mathcal{H}$ unknown?

Let us mention that in applications it is often possible to design the data space $\mathcal{X} \subset \mathbb{C}^L$ as well as c, while the channel space $\mathcal{H} \subset \mathcal{L}(\mathbb{C}^L, \mathbb{C}^L)$ is a priori fixed.

III. Necessary and sufficient conditions

Certainly, a naive approach to our problem is to first identify the channel $H \in \mathcal{H}$ and then to use the channel information to recover $x \in \mathcal{X}$. However, in principle it is not necessary to find the exact channel $H \in \mathcal{H}$ in order to recover $x \in \mathcal{X}$. In fact, we have the following necessary and sufficient condition for the recovery of x.

Proposition 2. *Let $\emptyset \neq \mathcal{H} \subset \mathcal{L}(\mathbb{C}^L, \mathbb{C}^L)$, $\emptyset \neq \mathcal{X} \subset \mathbb{C}^L$, and $c \in \mathbb{C}^L \backslash \{0\}$. Then every $x \in \mathcal{X}$ is uniquely recoverable from $y = H(x + c)$ with $H \in \mathcal{H} \backslash \{0\}$ unknown, if and only if*

($\bullet$) *$H(x + c) = H'(x' + c)$ for some $H, H' \in \mathcal{H} \backslash \{0\}$ and $x, x' \in \mathcal{X}$ implies $x = x'$.*

Proposition 2 gives a general solution to our problem. Unfortunately, the necessary and sufficient condition ($\bullet$) for the recovery of x is not practical and usually hard to verify. For this reason, in what follows we will derive some conditions that are easier to check.

For nonempty sets $\mathcal{H} \subset \mathcal{L}(\mathbb{C}^L, \mathbb{C}^L)$, $\mathcal{X} \subset \mathbb{C}^L$, and a vector $c \in \mathbb{C}^L \backslash \{0\}$, we define the following subsets of $\mathbb{C}^L$:

$$
\begin{aligned}
&\mathcal{H}c := \{Hc : H \in \mathcal{H}\},\\
&D(\mathcal{H}c) := \mathcal{H}c - \mathcal{H}c = \{(H - H')c : H, H' \in \mathcal{H}\},\\
&\mathcal{H}\mathcal{X} := \{Hx : H \in \mathcal{H},\ x \in \mathcal{X}\},\\
&D(\mathcal{H}\mathcal{X}) := \mathcal{H}\mathcal{X} - \mathcal{H}\mathcal{X},
\end{aligned}
$$

where 'D' stands for 'difference set'. In our setup, the vector c is fixed once it is chosen by the user; the sets $\mathcal{H}$ and $\mathcal{X}$ are not necessarily linear, so the sets defined above are not necessarily linear but satisfy

$$\mathcal{H}c,\, D(\mathcal{H}c) \subset \operatorname{span}\mathcal{H}c \quad \text{and} \quad \mathcal{H}\mathcal{X},\, D(\mathcal{H}\mathcal{X}) \subset \operatorname{span}\mathcal{H}\mathcal{X}.$$

Let us first consider two conditions that arise naturally from channel identification and signal recovery.

$$
\begin{aligned}
&(*)\ \ \text{The map } H \mapsto Hc \text{ is injective on } \mathcal{H},\\
&(**)\ \ \text{every } H \in \mathcal{H}\backslash\{0\} \text{ is injective on } \mathcal{X}.
\end{aligned}
$$

Condition $(*)$ is equivalent to the identifiability of $\mathcal{H}$ (see Definition 1), while condition $(**)$ is necessary and sufficient for the exact recovery of x provided that Hx is given.

Next, we consider the following condition, which, in case that $\mathcal{H}$ is linear, translates to having $y = H(x + c) \neq 0$ whenever $H \neq 0$.

$(\#)$ $H(x + c) = H'(x + c)$ for some $H, H' \in \mathcal{H}$ and $x \in \mathcal{X}$ implies $H = H'$.

If $\mathcal{H}$ contains at least two elements, this condition implies directly that $-c \notin \mathcal{X}$.

We also consider the following conditions.

(i) $\operatorname{span}\mathcal{H}c \cap \operatorname{span}\mathcal{H}\mathcal{X} = \{0\}$.

(ii) $D(\mathcal{H}c) \cap D(\mathcal{H}\mathcal{X}) = \{0\}$.

(iii) $H(\mathcal{X} + c) \cap H'(\mathcal{X} + c) = \emptyset$ for every $H \neq H'$ in $\mathcal{H}$.

(iv) $H(x + c) = H'(x' + c)$ for some $H \in \mathcal{H}\backslash\{0\}$, $H' \in \mathcal{H}$ and $x, x' \in \mathcal{X}$ implies $x = x'$.

(v) $\mathcal{H}c \cap \mathcal{H}\mathcal{X} = \{0\}$.

Theorem 3. *Let $\emptyset \neq \mathcal{H} \subset \mathcal{L}(\mathbb{C}^L, \mathbb{C}^L)$, $\emptyset \neq \mathcal{X} \subset \mathbb{C}^L$, and $c \in \mathbb{C}^L \backslash \{0\}$. Then*

$$
\begin{array}{ccccccccccc}
\text{(i)} & \Rightarrow & \text{(ii)} & \overset{(*)}{\Rightarrow} & \text{(iii)} & \underset{\substack{\mathcal{H}\ \textit{linear}\\ |\mathcal{X}|\geq 2,\\ \textit{or}\ (\#)}}{\overset{(**)}{\substack{\Rightarrow\\ \Leftarrow}}} & \text{(iv)} & \underset{(\#)}{\substack{\Rightarrow\\ \Leftarrow}} & (\bullet) & \Rightarrow & (**).\\
 & & & & \Downarrow & & \Downarrow{\scriptstyle \substack{\mathcal{H}\ \textit{linear}\\ 0\in\mathcal{X}}} & & \Downarrow{\scriptstyle \mathcal{H},\mathcal{X}\ \textit{linear}} & & \\
 & & & & (\#) & & \text{(v)} & & \text{(v)} & & \\
 & & & & \Downarrow{\scriptstyle 0\in\mathcal{X}} & & & & & & \\
 & & & & (*) & & & & & &
\end{array}
$$

In particular, if $\mathcal{H}$ is linear and $0 \in \mathcal{X}$, then condition (i) *together with $(*)$ and $(**)$ implies all other conditions, while condition* (iii) *and $(**)$ imply all other conditions except* (i) *and* (ii).

Remark 4. Assuming condition (i), one can immediately isolate Hx and Hc from the channel output $y = H(x{+}c) = Hx + Hc \in (\operatorname{span}\mathcal{H}\mathcal{X}) \oplus (\operatorname{span}\mathcal{H}c)$. Then $H \in \mathcal{H}$ can be identified from Hc by $(*)$, and in turn, x can be recovered uniquely from Hx by $(**)$.

Note that condition (iii) is precisely what is needed in order to identify the channel $H \in \mathcal{H}$ from the output $y = H(x{+}c)$ with unknown $x \in \mathcal{X}$. Once the channel H is identified, condition $(**)$ can be employed to recover x from $y' = y{-}Hc = Hx$.

On the other hand, neither condition (iv) nor condition $(\bullet)$ implies the exact recovery of $H \in \mathcal{H}$ but still guarantees the exact recovery of $x \in \mathcal{X}$.

Remark 5 (Degrees of freedom)**.** Suppose that both $\mathcal{H} \subset \mathcal{L}(\mathbb{C}^L, \mathbb{C}^L)$ and $\mathcal{X} \subset \mathbb{C}^L$ are linear subspaces with $\dim \mathcal{H}{=}k$ and $\dim \mathcal{X}{=}\ell$. By counting the degrees of freedom, we must have $k + \ell \leq L$, as a necessary condition for the exact recovery of $H \in \mathcal{H}$ and $x \in \mathcal{X}$ from $y = H(x{+}c)$. In fact, Theorem 3 implies that having exact recovery of $x \in \mathcal{X}$ necessitates condition (v) which can be expressed as

$$\mathcal{H}c \cap \bigcup_{H \in \mathcal{H}} H\mathcal{X} = \{0\},$$

where $H\mathcal{X} = \{Hx : x \in \mathcal{X}\} \subset \mathbb{C}^L$ is a linear subspace for each $H \in \mathcal{H}$. This implies $k + \ell \leq L$ if there exists some $H \in \mathcal{H}$ with $\ker H \cap \mathcal{X} = \{0\}$ so that $\dim H\mathcal{X} = \ell$.

On the other hand, condition (i) implies $k(\ell{+}1) \leq L$ which is certainly more than what is needed.

IV. Applications to Operator Paley–Wiener Spaces

Channel spaces $\mathcal{H} \subset \mathcal{L}(\mathbb{C}^L, \mathbb{C}^L)$ that are of special interest in modern communications are those spanned by a subset of time-frequency shift operators $T^k M^\ell$, $k, \ell = 0, \ldots, L-1$, where $T, M : \mathbb{C}^L \to \mathbb{C}^L$ are the cyclic translation (time shift) and modulation (frequency shift) operators defined by

$$
\begin{aligned}
&Tx = (x_{L-1}, x_0,\, x_1,\, \ldots,\, x_{L-2}),\\
&Mx = (\omega^0 x_0,\, \omega^1 x_1,\, \ldots,\, \omega^{L-1} x_{L-1}) \quad \text{with } \omega = e^{2\pi i/L}.
\end{aligned}
$$

Note that since $\{T^k M^\ell\}_{k,\ell=0}^{L-1}$ forms a basis of $\mathcal{L}(\mathbb{C}^L, \mathbb{C}^L)$, every $H \in \mathcal{L}(\mathbb{C}^L, \mathbb{C}^L)$ admits a unique representation

$$H = \textstyle\sum_{k,\ell=0}^{L-1} \eta_H(k,\ell)\, T^k M^\ell.$$

Restricting the support of η_H to a set $\Lambda \subset \mathbb{Z}_L{\times}\mathbb{Z}_L$ gives operators of the form

$$H = \textstyle\sum_{(k,\ell)\in\Lambda} \eta_H(k,\ell)\, T^k M^\ell \tag{1}$$

which constitute the following subspace of $\mathcal{L}(\mathbb{C}^L, \mathbb{C}^L)$.

Definition 6. *For $\Lambda \subset \mathbb{Z}_L{\times}\mathbb{Z}_L$, the* operator Paley–Wiener space[1] *is defined as*

$$
\begin{aligned}
OPW(\Lambda) &= \{H \in \mathcal{L}(\mathbb{C}^L, \mathbb{C}^L) : \operatorname{supp}\eta_H \subset \Lambda\}\\
&= \operatorname{span}\{T^k M^\ell : (k,\ell) \in \Lambda\}.
\end{aligned}
$$

[1] The terminology 'operator Paley–Wiener space' stems from the analogous identification problem in the continuous-time setting. There, $OPW(S)$ is the space of Hilbert–Schmidt operators on $L^2(\mathbb{R})$ whose spreading function is supported on $S \subset \mathbb{R}^2$ (equivalently, their Kohn–Nirenberg symbol is bandlimited to S). The results presented in this paper can be stated for the corresponding continuous-time operator Paley–Wiener spaces in a straightforward manner.

As we have $Hc = \sum_{(k,\ell)\in\Lambda} \eta_H(k,\ell)\, T^k M^\ell c$ for $H \in OPW(\Lambda)$ and $c \in \mathbb{C}^L$, the identifiability of $OPW(\Lambda)$ simply amounts to the linear independence of the vectors $\{T^k M^\ell c\}_{(k,\ell)\in\Lambda}$.

Definition 7. *The* Gabor matrix generated by a window $c \in \mathbb{C}^L$ *is the* $L\times L^2$ *matrix* $G(c)$ *consisting of time-frequency shifts* $T^k M^\ell c$, $k, \ell = 0, \dots, L-1$, *of* c *as columns, that is,* $G(c) = \{T^k M^\ell c\}_{k,\ell=0}^{L-1}$.

Theorem 8 ([6], [9])**.** *Given* $L \in \mathbb{N}$*, there exists a dense open subset* $\mathcal{S} \subset \mathbb{C}^L$ *of full measure such that for* $c \in \mathcal{S}$*, every* L *columns of* $G(c)$ *are linearly independent. Consequently, the space* $OPW(\Lambda)$ *with* $\Lambda \subset \mathbb{Z}_L\times\mathbb{Z}_L$ *is identifiable if and only if* $|\Lambda| \le L$.

Remark 9. When $\mathcal{H} = OPW(\Lambda)$ with $\Lambda \subset \mathbb{Z}_L\times\mathbb{Z}_L$, condition $(*)$ is equivalent to $G(c)|_\Lambda$ having linearly independent columns while condition $(\#)$ is equivalent to $G(x+c)|_\Lambda$ having linearly independent columns for every $x \in \mathcal{X}$.

As a consequence of Theorem 8, we have that for any $k \le L/2$ the nonlinear class

$$\Sigma_k = \{H \in \mathcal{L}(\mathbb{C}^L, \mathbb{C}^L) : |\operatorname{supp}\eta_H| \le k\} = \bigcup_{|\Lambda|=k} OPW(\Lambda)$$

is identifiable. This fact guides the study of identification of operators with unknown (or partially known) support [3], [7], [12].

A. Examples

For $L \in \mathbb{N}$, we denote by $\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_{L-1}$ the Euclidean basis vectors of $\mathbb{C}^L$.

Example 10. Let $L = 4$, $\Lambda = \{(0,0), (1,0)\}$ and $c = \boldsymbol{e}_0 = (1,0,0,0)$, so that $\mathcal{H} = OPW(\Lambda) = \operatorname{span}\{I, T\} \subset \mathcal{L}(\mathbb{C}^4, \mathbb{C}^4)$ and $\mathcal{H}c = D(\mathcal{H}c) = \operatorname{span}\mathcal{H}c = \mathbb{C}^2\times\{0\}^2$.
(a) If $x = u\,\boldsymbol{e}_2 = (0,0,u,0)$ for some $u \in \mathbb{C}$, then for any nontrivial $H \in OPW(\Lambda)$, say, $H = \alpha I + \beta T$ with $\alpha, \beta \in \mathbb{C}$, $(\alpha,\beta) \ne 0$, its response to the input $x+c$ is given by

$$y = H(x+c) = \alpha\boldsymbol{e}_0 + \beta\boldsymbol{e}_1 + \alpha u\boldsymbol{e}_2 + \beta u\boldsymbol{e}_3.$$

From the output y, one can directly read off the value of α and β, and in turn, u is determined exactly since we have either $\alpha \ne 0$ or $\beta \ne 0$ (or both).

In this case, we have $\mathcal{X} = \operatorname{span}\{\boldsymbol{e}_2\} = \operatorname{span}\{(0,0,1,0)\}$ and one can easily verify conditions $(*)$ and $(**)$. Also, we have $\mathcal{H}\mathcal{X} = D(\mathcal{H}\mathcal{X}) = \operatorname{span}\mathcal{H}\mathcal{X} = \{0\}^2\times\mathbb{C}^2$ which implies that condition (i) and therefore all other conditions hold by Theorem 3.

It should be noted that if x were transmitted directly through the channel H without first adding with c (alternatively, one may set $c = 0$), the exact recovery of x would have failed. Indeed, from the output $y = Hx = \alpha u\boldsymbol{e}_2 + \beta u\boldsymbol{e}_3$, we get $\alpha = y_2/u$, $\beta = y_3/u$ and any $u \in \mathbb{C}\backslash\{0\}$ if $(y_2, y_3) \ne (0,0)$; either any $\alpha, \beta \in \mathbb{C}$ and $u = 0$, or $\alpha = \beta = 0$ and any $u \in \mathbb{C}\backslash\{0\}$ if $y_2 = y_3 = 0$. Therefore, x cannot be recovered exactly. This shows the advantage of having the design freedom of c in our model. In fact, the design freedom of c is what allows us to distribute the degrees of freedom in the channel output y, therefore enabling the exact recovery of x.
(b) The arguments above, however, are not valid if x is of the form $x = u\,\boldsymbol{e}_0 = (u,0,0,0)$ with $u \in \mathbb{C}$, because then

$$y = H(x+c) = \alpha(u+1)\boldsymbol{e}_0 + \beta(u+1)\boldsymbol{e}_1$$

determines α and β only up to a common scale factor $(u+1) \in \mathbb{C}$ (hence, u can take any value in $\mathbb{C}$ if $\alpha = \beta = 0$; any value in $\mathbb{C}$ except -1 otherwise).

In this case, we have $\mathcal{X} = \operatorname{span}\{\boldsymbol{e}_0\} = \operatorname{span}\{(1,0,0,0)\}$ and one can verify conditions $(*)$ and $(**)$ as in (a). However, we have $\operatorname{span}\mathcal{H}\mathcal{X} = \{0\}^2\times\mathbb{C}^2$, so that condition (v) and therefore conditions (i)–(iv) do not hold by Theorem 3. Also, one can check that conditions $(\#)$ and $(\bullet)$ do not hold.
(c) Let us consider the nonlinear set

$$\mathcal{X} = \operatorname{span}\{(0,1,1,0)\} \cup \operatorname{span}\{(0,1,-1,0)\}.$$

Then $\mathcal{H}\mathcal{X} = \operatorname{span}\{(0,1,1,0), (0,0,1,1)\} \cup \operatorname{span}\{(0,1,-1,0), (0,0,1,-1)\}$ and $D(\mathcal{H}\mathcal{X}) = \operatorname{span}\mathcal{H}\mathcal{X} = \{0\}\times\mathbb{C}^3$, which shows that conditions (i) and (ii) fail but condition (v) holds. It is easy to verify conditions $(*)$, $(**)$ and $(\#)$. Conditions (iii), (iv) and $(\bullet)$ do not hold, since we have e.g., $H(x+c) = H'(x'+c)$ with $H = 2I + (1+\sqrt{5})T$, $H' = 2I + (3-\sqrt{5})T$, $x = (0, 2-\sqrt{5}, 2-\sqrt{5}, 0)$ and $x' = (0,1,-1,0)$. In summary, conditions $(*)$, $(**)$, $(\#)$ and (v) hold while conditions (i)–(iv) and $(\bullet)$ do not.

Note that even though we have $\mathcal{H}c + \mathcal{H}\mathcal{X} = \mathbb{C}^4$ and $\mathcal{H}c \cap \mathcal{H}\mathcal{X} = \{0\}$, the method described in Remark 4 does not work because each vector (y_0, y_1, y_2, y_3) in $\mathbb{C}^4$ admits two different decompositions with components in $\mathcal{H}c$ and $\mathcal{H}\mathcal{X}$: $(y_0, y_1, y_2, y_3) = (y_0, y_1 - y_2 + y_3, 0, 0) + [(0, y_2 - y_3, y_2 - y_3, 0) + (0, 0, y_3, y_3)] = (y_0, y_1 + y_2 + y_3, 0, 0) + [(0, -y_2 - y_3, y_2 + y_3, 0) + (0, 0, -y_3, y_3)]$.

We also give an example where both $\mathcal{H} \subset \mathcal{L}(\mathbb{C}^L, \mathbb{C}^L)$ and $\mathcal{X} \subset \mathbb{C}^L$ are linear and all conditions except (i) and (ii) hold.

Example 11. Let $L = 8$, $\Lambda = \{(0,0), (1,0), (2,0)\}$ and $c = \boldsymbol{e}_5$, so that $\mathcal{H} = OPW(\Lambda) = \operatorname{span}\{I, T, T^2\} \subset \mathcal{L}(\mathbb{C}^8, \mathbb{C}^8)$ and $\mathcal{H}c = \operatorname{span}\{\boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7\} = \{0\}^5\times\mathbb{C}^3$. Let $\mathcal{X} = \operatorname{span}\{\boldsymbol{e}_0, \boldsymbol{e}_2+\boldsymbol{e}_3+\boldsymbol{e}_4\} \subset \mathbb{C}^8$. Then $D(\mathcal{H}\mathcal{X})$ contains $\boldsymbol{e}_5$ and $\operatorname{span}\mathcal{H}\mathcal{X} = \operatorname{span}\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3+\boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_4+\boldsymbol{e}_6\}$, which shows that conditions (i) and (ii) fail. However, one can easily verify conditions $(**)$ and (iii), hence, Theorem 3 implies that all conditions except (i) and (ii) hold.

Remark 12 (Relations with blind deconvolution)**.**
(a) If $\mathcal{H} = \operatorname{span}\{I, T, \dots, T^{L-1}\}$, then every $H \in \mathcal{H}$ can be expressed in the form $H = a_0 I + a_1 T + \dots + a_{L-1}T^{L-1}$ and therefore with $z := x + c$ we have $y = H(x+c) = a * z$, where $a * z \in \mathbb{C}^L$ is the discrete circular convolution of $a = \{a_\ell\}_{\ell=0}^{L-1}$ and z defined by $(a*z)_\ell = \sum_{k=0}^{L-1} a_k\, z_{(\ell-k)_{\mathrm{mod}\ L}}$ for $\ell = 0, \dots, L-1$. This reduces our setup to the framework of blind deconvolution [1], [4], [8], [13] where the goal is to recover a and z simultaneously by observing y. Certainly, to make the problem feasible one has to restrict H further to a subset of $\mathcal{H}$ and also z to a subset $\mathcal{Z} \subset \mathbb{C}^L$. Note that $\mathcal{Z} = c + \mathcal{X}$ in our setup.
(b) The case $\mathcal{H} = \operatorname{span}\{I, M, \dots, M^{L-1}\}$ can be interpreted

in a similar way by replacing the canonical basis of $\mathbb{C}^L$ with the Fourier basis of $\mathbb{C}^L$. Moreover, a similar interpretation holds for $\mathcal{H} = \operatorname{span}\{I, TM^s, \dots, T^{L-1}M^{(L-1)s}\}$ with $s \in \{1,\dots,L-1\}$ and $L \in \mathbb{N}$ odd, as one can find a basis of $\mathbb{C}^L$ that diagonalizes the operators $I, TM^s, \dots, T^{L-1}M^{(L-1)s}$ simultaneously.

(c) In the cases discussed above, the generators of $\mathcal{H}$ commute, that is, for any $s \in \{0,1,\dots,L-1\}$ we have $(T^kM^{ks})(T^{k'}M^{k's}) = (T^{k'}M^{k's})(T^kM^{ks})$, $0 \le k,k' \le L-1$, and also $M^kM^{k'} = M^{k'}M^k$, $0 \le k,k' \le L-1$. The analysis becomes more involved when $\mathcal{H}$ contains generators that do not commute, e.g., if $\mathcal{H}$ contains T and M at the same time.

V. Some Relevant Properties of Gabor Matrices

As discussed in Remark 4, condition (i) implies that $\operatorname{span}\mathcal{H}c + \operatorname{span}\mathcal{H}\mathcal{X} \subset \mathbb{C}^L$ is the direct sum of $\operatorname{span}\mathcal{H}c$ and $\operatorname{span}\mathcal{H}\mathcal{X}$, and therefore each of its elements admits a unique decomposition into components in $\operatorname{span}\mathcal{H}c$ and $\operatorname{span}\mathcal{H}\mathcal{X}$. This allows a unique separation of Hc and Hx from the channel output $y = H(x+c)$.

Motivated by this observation, we deviate from our main problem and ask the following question: given $\mathcal{H} \subset \mathcal{L}(\mathbb{C}^L, \mathbb{C}^L)$ and a subspace $\widetilde{\mathcal{M}} \subset \mathbb{C}^L$, does there exist a vector $c \in \mathbb{C}^L\backslash\{0\}$ with $(\operatorname{span}\mathcal{H}c) \cap \widetilde{\mathcal{M}} = \{0\}$?

In case that $\mathcal{H} = OPW(\Lambda)$ with $\Lambda \subset \mathbb{Z}_L \times \mathbb{Z}_L$, we have $\operatorname{span}\mathcal{H}c = \operatorname{ran} G(c)|_\Lambda$ and therefore the question leads to investigation of properties of Gabor matrices.

Lemma 13. *Let $\Lambda \subset \mathbb{Z}_L \times \mathbb{Z}_L$ with $1 \le |\Lambda| \le L-1$. Then*
$$\operatorname{span}\big\{\ker(G(c)|_\Lambda)^* : c \in \mathcal{S}\big\} = \mathbb{C}^L,$$
where $\mathcal{S} \subset \mathbb{C}^L$ is the set in Theorem 8.

Theorem 14. *Let $\Lambda \subset \mathbb{Z}_L \times \mathbb{Z}_L$ with $1 \le |\Lambda| \le L-1$ and $a \in \mathbb{C}^L\backslash\{0\}$. There exists a vector $c \in \mathbb{C}^L\backslash\{0\}$ such that the matrix $[\,G(c)|_\Lambda,\, a\,] \in \mathbb{C}^{L\times(|\Lambda|+1)}$ has full rank.*

It is unclear whether this result extends to the case of multiple vectors $a^{(1)}, \dots, a^{(N)}$.

Now, let us consider the simplest case where $\mathcal{X} \subset \mathbb{C}^L$ is a subspace of dimension 1, that is, $\mathcal{X} = \operatorname{span}\{z\}$ for some $z \in \mathbb{C}^L\backslash\{0\}$. In this case the task of recovering $x = uz$ with $u \in \mathbb{C}$ is equivalent to recovering its coefficient $u \in \mathbb{C}$. With $H \in OPW(\Lambda)$ expressed in the form (1), the channel output $y = H(x+c)$ reads
$$\begin{aligned} y &= u \sum_{(k,\ell)\in\Lambda} \eta_H(k,\ell)\, T^kM^\ell z + \sum_{(k,\ell)\in\Lambda} \eta_H(k,\ell)\, T^kM^\ell c \\ &= u\,(G(z)|_\Lambda\, \eta) + G(c)|_\Lambda\, \eta, \end{aligned}$$
where $\eta = \{\eta_H(k,\ell)\}_{(k,\ell)\in\Lambda}$. As our purpose is to recover $u \in \mathbb{C}$ (and if possible, to recover η while doing so), it is desirable to have a small dimension for $\operatorname{ran} G(z)|_\Lambda$ so as to reserve enough space for $\operatorname{ran} G(c)|_\Lambda$ (recall that $G(c)|_\Lambda$ must have full column rank for the exact recovery of η from $y' = G(c)|_\Lambda\, \eta$). For example, if $\operatorname{ran} G(z)|_\Lambda$ has dimension 1 and $|\Lambda| = L-1$ (e.g., take $z = (1,1,\dots,1)$ and $\Lambda = \{(0,0),(1,0),\dots,(L-2,0)\}$), then according to Theorem 14 we can pick $c \in \mathbb{C}^L\backslash\{0\}$ so that $(\operatorname{ran} G(z)|_\Lambda) \oplus (\operatorname{ran} G(c)|_\Lambda) = \mathbb{C}^L$. This allows us to separate Hx and Hc from $y = H(x+c)$, where Hc is then used to identify H, and in turn, $u \in \mathbb{C}$ is recovered from $y'' = Hx = u(Hz)$ provided that $Hz \ne 0$.

The discussion above leads to the following question: given a set $\Lambda \subset \mathbb{Z}_L \times \mathbb{Z}_L$ with $|\Lambda| \le L$, what is the minimum rank of $G(z)|_\Lambda$ for z varying in $\mathbb{C}^L\backslash\{0\}$? Note that the maximum rank of $G(z)|_\Lambda$ is $|\Lambda|$ by Theorem 8.

Proposition 15. Let $L \ge 3$ be an odd integer and $\Lambda \subset \mathbb{Z}_L \times \mathbb{Z}_L$ with $|\Lambda| \le L$. Then
$$\min_{z\in\mathbb{C}^L\backslash\{0\}} \operatorname{rank}(G(z)|_\Lambda) \;\le\; N(\Lambda), \tag{2}$$
where $N(\Lambda)$ is defined by
$$\min_{s\in\{0,\dots,L-1,\infty\}} \min\big\{|I| : I \subset \mathbb{Z}_L \times \mathbb{Z}_L \text{ with } \Lambda \subset I + \Gamma_s\big\}$$
with
$$\Gamma_s = \big\{(0,0),(1,s),\dots,(L-1,(L-1)s)\big\}, \;\; 0 \le s \le L-1,$$
$$\Gamma_\infty = \big\{(0,0),(0,1),\dots,(0,L-1)\big\},$$
and $I + \Gamma_s = \{x + y : x \in I,\ y \in \Gamma_s\}$.

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