The Solvability Complexity Index of Sampling-based Hilbert Transform Approximations

Holger Boche†* and Volker Pohl†
† Chair of Theoretical Information Technology, Technical University of Munich, 80 333 München, Germany
* Munich Center for Quantum Science and Technology (MCQST), 80 799 München, Germany

***Abstract*—This paper determines the solvability complexity index (SCI) and a corresponding tower of algorithms for the computational problem of calculating the Hilbert transform of a continuous function with finite energy from its samples. It is shown that the SCI of these algorithms is equal to 2 and that the SCI is independent on whether the calculation is done by linear or by general (i.e. linear and/or non-linear) algorithms.**

***Index Terms*—Computational complexity for continuous problems, Hilbert transform, sampling, towers of algorithms**

I. Introduction

Many problems in science and engineering are such that they can generally not be computed exactly in finitely many steps, i.e. in finite time. Well known examples include the determination of eigenvalues of matrices, the determination of roots of polynomials, the calculation of solutions of certain differential equations, or the recovery of bandlimited signals from their samples. If the solution f of such a computational problem can not be obtained in finite time, one asks for an algorithm which determines a sequence $\{f_n\}_{n\in\mathbb{N}}$ of approximations of f, wherein each f_n can be computed in finite time and such that for every arbitrary ϵ, one can determine an $N_0 \in \mathbb{N}$ such that $\|f - f_n\| < \epsilon$ for all $n \geq N_0$ with a norm $\|\cdot\|$ determined by the concrete problem to be solved. Then the solution f can be computed up to any arbitrarily small error ϵ in finite time and f would be called *computable* or *approximable* in the above sense. The classical theory of computations and complexity provides a rich mathematical framework to classify and investigate the complexity of such approximation processes.

However, it is known that there are several computational problems which can not be solved by determining the limit of only *one* sequence of approximations but one may need *several* limits to compute the solution. Example can be found in the theory of computable functions [1], [2] and Turing jumps which is part of classical recursion theory [3] and which is known in mathematical logic as arithmetical (or Kleene–Mostowski) hierarchy [4], [5]. Another well known example are the McMullen–Doyle towers [6], [7] for the polynomial root finding problem. So these examples show that there are problems which might be solvable by determining the limit of an approximation sequences $f_{n_1,\dots,n_k}$ which depend on several indices $n_1, \dots, n_k$, and by passing $n_1, \dots, n_k$ to infinity (in an appropriated order). A general approach to analyze such multi-parameter approximation problems was proposed recently in the very interesting papers [8], [9]. This framework formalizes the known approaches, mentioned above, to full generality. Therein, the so called *Solvability Complexity Index (SCI)* of a computational problem with respect to a particular class of algorithms, is the smallest number of limits needed to compute the solution of the problem. This way, a powerful classification of the complexity of algorithms for computational problems was obtained.

This paper uses the SCI framework to determine the solvability complexity index of sampling based algorithms for the determination of the Hilbert transform, an operation which plays a fundamental role in many different areas of science and engineering because of its close relation to causal physical processes [10], [11]. In a series of papers [12], [13], [14], it was shown that on a large class of continuous functions of finite energy, there exists no sampling based algorithm which can determine the Hilbert transform for all functions in this class. We are going to show that this result is equivalent to saying that the SCI of sampling based Hilbert transform approximations is larger than 1, and we will show that the SCI of these algorithms is exactly equal to 2. Moreover, it is shown that this value of the SCI does even not depend on the class of algorithms which is considered. For linear as well as for general (i.e. not necessarily linear) algorithms, the SCI is always equal to 2.

The paper is organized as follows. At the beginning, Section II defines the family of signal spaces on which the sampling based Hilbert transform approximations are investigated. After recalling a basic result concerning the convergence of Hilbert transform approximations in Sect. III, Section IV formulates the problem of calculating the Hilbert transform in the SCI framework. Then Section V determines the SCI of such algorithms and derives a corresponding tower of algorithms. The paper closes with a short discussion in Section VI.

II. Notations and Signal Spaces

This section clarifies the main notations and it introduces the Banach spaces on which the Hilbert transform is investigated.

A. Basic definitions and notations

Throughout this paper, we consider continuous functions f on the interval $\mathbb{T} = [0, 2\pi]$ with $f(0) = f(2\pi)$. The set of all these functions is denoted by $\mathcal{C}$ and equipped with the norm $\|f\|_{\mathcal{C}} = \max_{t\in\mathbb{T}} |f(t)|$. For every $f \in \mathcal{C}$, its *Fourier coefficients* $c_k(f)$ are well defined by

$$c_k(f) = \tfrac{1}{2\pi} \int_{\mathbb{T}} f(t)\, \mathrm{e}^{-\mathrm{i}kt}\, \mathrm{d}t\,, \qquad k = 0, \pm 1, \pm 2, \dots\,. \tag{1}$$

Let $f \in \mathcal{C}$ be arbitrary with Fourier coefficients (1). Then for every $n \in \mathbb{N}$, the *partial Fourier series* of f is defined by

$$(\mathrm{P}_n f)(t) = \textstyle\sum_{k=-n}^{n} c_k(f)\, \mathrm{e}^{\mathrm{i}kt}\,, \qquad t \in \mathbb{T}\,.$$

Moreover, with f we associate its *conjugate function*, defined by

$$\widetilde{f}(t) = (\mathrm{H}f)(t) = -\mathrm{i} \textstyle\sum_{k\in\mathbb{Z}} \mathrm{sgn}(k)\, c_k(f)\, \mathrm{e}^{\mathrm{i}kt}\,, \qquad t \in \mathbb{T} \tag{2}$$

provided this sum converges is some sense and with the sign function $\mathrm{sgn}(n) = n/|n|$ for $n \neq 0$ and $\mathrm{sgn}(0) = 0$. The mapping $\mathrm{H} : f \mapsto \widetilde{f}$ is also known as the *Hilbert transform*. Form this definition, it is clear that $c_k(\widetilde{f}) = -\mathrm{i}\,\mathrm{sgn}(k) c_k(f)$ and in particular $\mathrm{P}_n\mathrm{H}f = \mathrm{HP}_n f$.

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B. Signal spaces of finite energy

Let $f \in \mathcal{C}$ be arbitrary. Based on its Fourier coefficients, we define for any $\beta \geq 0$ the seminorm

$$\|f\|_{\beta} = \left(\sum_{k \in \mathbb{Z},\ k \neq 0} |k| \left(1 + \log|k|\right)^{\beta} |c_k(f)|^2 \right)^{1/2} \tag{3}$$

and therewith the linear space

$$\mathcal{B}_{\beta} = \left\{ f \in \mathcal{C} \ : \ \widetilde{f} \in \mathcal{C} \text{ and } \|f\|_{\beta} < \infty \right\} . \tag{4}$$

Equipped with the norm

$$\|f\|_{\mathcal{B}_{\beta}} = \max\left(\|f\|_{\mathcal{C}}, \|\widetilde{f}\|_{\mathcal{C}}, \|f\|_{\beta} \right) ,$$

the space $\mathcal{B}_{\beta}$ becomes a Banach space, the primary signal space in the following investigations. It is useful to note that the definition of the seminorm (3) implies $\|f\|_{\mathcal{B}_{\beta}} = \|\widetilde{f}\|_{\mathcal{B}_{\beta}}$ for all $f \in \mathcal{B}_{\beta}$.

For $\beta = 0$, the seminorm (3) characterizes the *Dirichlet energy* of the function f and the set $\{f \in \mathcal{C} : \|f\|_{\beta=0} < \infty\}$ corresponds to the well known Sobolev space $H^{1/2}(\mathbb{T}) = W^{1/2,2}(\mathbb{T})$. Moreover, in the scale of the Banach space $\{\mathcal{B}_{\beta}\}_{\beta \geq 0}$, the parameter β characterizes the concentration of the (Dirichlet) energy in the Fourier coefficients. As larger β as faster decay the amplitude of the Fourier coefficients $c_k(f)$ as $k \to \infty$. In particular, we have the relation $\mathcal{B}_{\beta'} \subset \mathcal{B}_{\beta} \subset \mathcal{B}_0$ for all $\beta' \geq \beta \geq 0$ and we refer to [14], [15] for a more detailed discussion on the importance and the properties of the spaces $\mathcal{B}_{\beta}$ and there relation to certain physical problems.

It is well known that for arbitrary continuous functions $f \in \mathcal{C}$ the partial Fourier series $\mathrm{P}_n f$ does generally not converge to f in the norm of $\mathcal{C}$. Nevertheless, it is not hard to see that for all $f \in \mathcal{B}_{\beta}$ the situation is much better.

Lemma 1: *For every $\beta \geq 0$ and every $f \in \mathcal{B}_{\beta}$ holds*

$$\lim_{n \to \infty} \left\| f - \mathrm{P}_n f \right\|_{\mathcal{C}} = 0 \quad \textit{and} \quad \lim_{n \to \infty} \left\| \widetilde{f} - \mathrm{P}_n \widetilde{f} \right\|_{\mathcal{C}} = 0 .$$

We refer to [14, Lemma 1] for a proof of this simple statement.

III. The non-computability of the Hilbert transform

The Hilbert transform (2), also known as *Kramers–Kronig relation*, plays a central role in physic and engineering because of its close relation to the concept causality [10], [11]. Especially in engineering, there are many applications where it is important to calculate the Hilbert transform of a given (or measured) function f. Nowadays, such calculations are (almost) always done on digital computers. Therefore, the question arises whether there exists an algorithm which is able to approximate $\mathrm{H}f$ arbitrarily well from finitely many samples of f for all functions from a certain signal space.

The described problem is just one particular example of sampling based signal processing methods which are the basis of modern signal processing, especially in a period of an increasing digitization in science and technology. The problem of finding sampling based algorithms for the computation of the Hilbert transform was investigated by the authors in a series of papers [12], [13], [14], [16] and the following general statement could finally be proved.

Theorem 2 (Corollary 3.2 in [16]): *Let $\{\Gamma_n\}_{n \in \mathbb{N}}$ be a sequence of lower semicontinuous mappings $\Gamma_n : \mathcal{B}_{\beta} \to \mathcal{C}$ so that to every $n \in \mathbb{N}$ there exists a finite subset $\mathcal{T}_n \subset \mathbb{T}$ such that $f_1(\tau) = f_2(\tau)$ for all $\tau \in \mathcal{T}_n$ implies $\Gamma_n(f_1) = \Gamma_n(f_2)$. Then for every $0 \leq \beta \leq 1$ there exist functions $f \in \mathcal{B}_{\beta}$ such that $\limsup_{n \to \infty} \|\mathrm{H}f - \Gamma_n(f)\|_{\mathcal{C}} > 0$.*

Remark 1: It was also shown that for every $\beta > 1$ there is a sequence $\{\Gamma_n\}_{n \in \mathbb{N}}$ of *linear* mappings $\Gamma_n : \mathcal{B}_{\beta} \to \mathcal{C}$ with the properties from Theorem 2 and such that $\lim_{n \to \infty} \|\mathrm{H}f - \Gamma_n f\|_{\mathcal{C}} = 0$ for all $f \in \mathcal{B}_{\beta}$.

Theorem 2 is the motivation for our following investigations. As we will see in Theorem 3 below, Theorem 2 implies that the SCI of sampling based Hilbert transform approximations on $\mathcal{B}_{\beta}$ has to be larger than 1. So it is an interesting problem to determine the SCI of this particular computational problem. Since the previous remark implies that on the spaces $\mathcal{B}_{\beta}$ with $\beta > 0$, the SCI of the Hilbert transform is at most 1, we consider subsequently only Hilbert transform approximations on spaces $\mathcal{B}_{\beta}$ with $\beta \in [0, 1]$.

IV. Towers of Algorithms and the SCI

We follow the general framework for computational problems introduced in [8], [9]. The basic objects in this framework are

- Ω : the *primary set*, i.e. a set of objects
- Λ : the *evaluation set*, i.e. a set of functionals on Ω
- $\mathcal{M}$: a metric space
- Ξ : the *problem function*, i.e. a mapping $\Xi : \Omega \to \mathcal{M}$

Because of space constraints, we do not explain the framework in its full generality but we apply this framework directly to our concrete problem. In our case, the above defined objects are given as follows.

- $\Omega = \mathcal{B}_{\beta}$ for some $\beta \in [0, 1]$.
- Λ is the set of functionals $\mathrm{S}_t : \mathcal{B}_{\beta} \to \mathbb{C}$ with $t \in \mathbb{T}$ defined by
$$\mathrm{S}_t(f) = f(t) , \quad f \in \mathcal{B}_{\beta} .$$
- $\mathcal{M}$ will be either $\mathcal{C}(\mathbb{T})$ or $\mathcal{B}_{\beta}$, i.e. even a normed space.
- $\Xi = \mathrm{H} : \mathcal{B}_{\beta} \to \mathcal{M}$ is the Hilbert transform (2).

Remark 2: Both choices $\mathcal{M} = \mathcal{C}$ and $\mathcal{M} = \mathcal{B}_{\beta}$ are of interest in practical applications [14].

Definition 1 (Computational Problem): *We call the collection $\{\mathrm{H}, \mathcal{B}_{\beta}, \mathcal{M}, \Lambda\}$ the* computational problem *(of sampling based Hilbert transform approximations).*

In other words, the particular computational problem studied in this paper is as follows. We consider functions in the primary set (i.e. the signal space) $\mathcal{B}_{\beta}$ of continuous functions with finite energy. The evaluation set Λ models our signal acquisition. Here, the possible acquisition functionals are point evaluations of the signals $f \in \mathcal{B}_{\beta}$ at points $t \in \mathbb{T}$. Then the goal is to determine an approximation of the problem function (i.e. the Hilbert transform H) for all functions from the primary set but based on a finite subset of the samples $\{\mathrm{S}(f)\}_{\mathrm{S} \in \Lambda}$. The quality of the approximation is measured in the metric of $\mathcal{M}$ (i.e. in the norm of $\mathcal{C}$ or $\mathcal{B}_{\beta}$).

The problem of calculating $\mathrm{H}f$ from samples of $f \in \mathcal{B}_{\beta}$ will be solved by a tower of algorithms. The following definition characterizes the fundamental algorithms which will build the lowest (fundamental) level of the tower of algorithms.

Definition 2 (Fundamental Algorithm): *Let $\beta \in [0, 1]$ be arbitrary and let $\{\mathrm{H}, \mathcal{B}_{\beta}, \mathcal{M}, \Lambda\}$ be a computational problem of sampling based Hilbert transform approximations. A* fundamental algorithm *is a mapping $\Gamma : \mathcal{B}_{\beta} \to \mathcal{M}$ such that*

1) *there exists a finite subset $\Lambda_{\Gamma} \subset \Lambda$,*
2) *for every $f \in \mathcal{B}_{\beta}$ the action of Γ on f depends only on the values $\{\mathrm{S}(f)\}_{\mathrm{S} \in \Lambda_{\Gamma}}$,*
3) *if some $f, g \in \mathcal{B}_{\beta}$ satisfy $\mathrm{S}(g) = \mathrm{S}(f)$ for all $\mathrm{S} \in \Lambda_{\Gamma}$ then $\Gamma(f) = \Gamma(g)$,*
4) *if $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{B}_{\beta}$ is a sequence with $\lim_{n \to \infty} \|f - f_n\|_{\mathcal{B}_{\beta}} = 0$ then $\lim_{n \to \infty} \|\Gamma(f) - \Gamma(f_n)\|_{\mathcal{M}} = 0$.*

We say that Γ is a fundamental algorithm of type L, *if Γ is linear. Γ is said to be of* type G, *if there is no restriction on linearity of Γ.*

Remark 3: Note the close relation between Properties 1)–3) in this definition and the requirements on the sequence $\{\Gamma_n\}_{n \in \mathbb{N}}$ in Theorem 2. Note also that Property 4) in Definition 2 is not contained in the general framework from [9] but it is needed here to apply later Theorem 2. It requires that the fundamental algorithms are (sequential) continuous mappings $\mathcal{B}_\beta \to \mathcal{M}$.

According to Definition 2, we broadly distinguish between fundamental algorithms of type *linear* (L) and of type *general* (G). Type G means that we make no restriction on the algorithms. They can be either linear or non-linear. Note also that if Γ is of type L then it will have a fairly simple structure. Indeed, assume $\{t_k\}_{k=1}^n \subset \mathbb{T}$ is the finite sampling set which defines the evaluation set $\Lambda_\Gamma = \{\mathrm{S}_k\}_{k=1}^n \subset \Lambda$ by $\mathrm{S}_k(f) = f(t_k)$. Then the linearity of Γ implies that it has the form

$$(\Gamma f)(t) = \sum_{k=1}^n \mathrm{S}_k(f)\, \gamma_k(t) = \sum_{k=1}^n f(t_k)\, \gamma_k(t)$$

for all $t \in \mathbb{T}$ and with functions $\{\gamma_k\}_{k=1}^n \subset \mathcal{M}$ which completely determine Γ. Moreover, this representation of Γ implies immediately that it is also bounded.

The calculation of $\mathrm{H}f$ is not entirely performed by fundamental algorithms but by a tower of algorithms. These are families of sequences of algorithms where the operators in a certain level are obtain as a limit of algorithms from a lower level. In the top level of the algorithm, $\mathrm{H}f$ is finally calculated.

Definition 3 (Tower of Algorithms): *Given the computational problem* $\{\mathrm{H}, \mathcal{B}_\beta, \mathcal{M}, \Lambda\}$ *for some* $\beta \in [0,1]$. *A* tower of algorithms *of type* α *of height* k *for* $\{\mathrm{H}, \mathcal{B}_\beta, \mathcal{M}, \Lambda\}$ *is a family of sequences of operators*

$$\begin{aligned} \Gamma_{n_k} &: \mathcal{B}_\beta \to \mathcal{M}\,, \\ \Gamma_{n_k, n_{k-1}} &: \mathcal{B}_\beta \to \mathcal{M}\,, \\ &\vdots \\ \Gamma_{n_k, n_{k-1}, \dots, n_1} &: \mathcal{B}_\beta \to \mathcal{M}\,, \end{aligned}$$

where $n_k, \dots, n_1 \in \mathbb{N}$ *and where the operators* $\Gamma_{n_k, n_{k-1}, \dots, n_1}$ *at the lowest level of the tower are fundamental algorithms of type* α *in the sense of Def. 2. Moreover, for every* $f \in \mathcal{B}_\beta$ *holds*

$$\begin{aligned} \mathrm{H}(f) &= \lim_{n_k \to \infty} \Gamma_{n_k}(f) \\ \Gamma_{n_k}(f) &= \lim_{n_{k-1} \to \infty} \Gamma_{n_k, n_{k-1}}(f) \\ &\vdots \\ \Gamma_{n_k, \dots, n_2}(f) &= \lim_{n_1 \to \infty} \Gamma_{n_k, \dots, n_2, n_1}(f) \end{aligned}$$

where the convergence is always in the metric of $\mathcal{M}$.

Remark 4: Note that in a tower of algorithms of type L, all operators, in each level of the tower, are linear and bounded. Since the operators in the lowest level are linear and bounded, it is easily seen that the operators in the higher levels are also linear and bounded. A tower of type $\alpha = G$ contains arbitrary operators.

The height of a tower of algorithms is an important parameter since it characterizes in some sense the complexity of the corresponding computational problem. More precisely, given a computational problem $\{\mathrm{H}, \mathcal{B}_\beta, \mathcal{M}, \Lambda\}$, we are interested in finding a tower of algorithms whose height is as small as possible.

Definition 4 (Solvability Complexity Index – SCI): *Given a computational problem* $\{\mathrm{H}, \mathcal{B}_\beta, \mathcal{M}, \Lambda\}$. *We say that* $\{\mathrm{H}, \mathcal{B}_\beta, \mathcal{M}, \Lambda\}$ *has a* Solvability Complexity Index $\mathrm{SCI}\left(\mathrm{H}, \mathcal{B}_\beta, \mathcal{M}, \Lambda\right)_\alpha = k$ *with respect to towers of algorithms of type* α *if* k *is the smallest integer for which there exists a tower of algorithms of type* α *of height* k. *If no such tower exists then* $\mathrm{SCI}\left(\mathrm{H}, \mathcal{B}_\beta, \mathcal{M}, \Lambda\right)_\alpha = \infty$.

If the SCI of a certain computational problem is $k < \infty$, it means that the problem can be computed by k limiting processes. Clearly, the larger k the more demanding is the computation. A SCI of zero would mean that the problem can be computed in finitely many steps. In our case, the SCI $\left(\mathrm{H}, \mathcal{B}_\beta, \mathcal{M}, \Lambda\right)_\alpha$ depends generally only on the two parameters $\mathcal{M}$ and α, because the problem function, the primary set, and the evaluation set are fixed in our setting. So to simplify notations, we write sometimes $\mathrm{SCI}_\mathrm{H}\left(\mathcal{M}\right)_\alpha$ instead of $\mathrm{SCI}\left(\mathrm{H}, \mathcal{B}_\beta, \mathcal{M}, \Lambda\right)_\alpha$.

V. The SCI Index of the Hilbert transform

This section investigate the SCI of sampling based algorithms for calculating the Hilbert transform of functions in $\mathcal{B}_\beta$. So for an arbitrary $\beta \in [0,1]$, we want to determine the SCI of the computational problem defined in Def. 1. In principle, we consider the two computational problems

$$\{\mathrm{H}, \mathcal{B}_\beta, \mathcal{C}, \Lambda\} \qquad \text{and} \qquad \{\mathrm{H}, \mathcal{B}_\beta, \mathcal{B}_\beta, \Lambda\}$$

and we look for towers of algorithms of type $\alpha = L$ (linear) and of type G (general). First we notice that because the set of all linear operators is a subset of all general operators, one has the relation

$$\mathrm{SCI}\left(\mathrm{H}, \mathcal{B}_\beta, \mathcal{M}, \Lambda\right)_G \leq \mathrm{SCI}\left(\mathrm{H}, \mathcal{B}_\beta, \mathcal{M}, \Lambda\right)_L \tag{5}$$

for $\mathcal{M} = \mathcal{C}$ or $\mathcal{M} = \mathcal{B}_\beta$. Moreover, since $\mathcal{B}_\beta$ is continuously embedded in $\mathcal{C}$ with $\|f\|_\mathcal{C} \leq \|f\|_{\mathcal{B}_\beta}$ for all $f \in \mathcal{B}_\beta$, we have for both types $\alpha = L$ and $\alpha = G$ also

$$\mathrm{SCI}\left(\mathrm{H}, \mathcal{B}_\beta, \mathcal{C}, \Lambda\right)_\alpha \leq \mathrm{SCI}\left(\mathrm{H}, \mathcal{B}_\beta, \mathcal{B}_\beta, \Lambda\right)_\alpha \tag{6}$$

because if a tower of algorithm converges in $\mathcal{B}_\beta$, it converges a fortiori in $\mathcal{C}$.

Our main contribution is the following complete characterization of the SCI for the computational problem of sampling based Hilbert transform approximations.

Theorem 3: *Let* $\beta \in [0,1]$ *be arbitrary, then we have*

$$\mathrm{SCI}\left(\mathrm{H}, \mathcal{B}_\beta, \mathcal{B}_\beta, \Lambda\right)_L = \mathrm{SCI}\left(\mathrm{H}, \mathcal{B}_\beta, \mathcal{C}, \Lambda\right)_G = 2\,.$$

Remark 5: Theorem 3 implies in particular that for both Banach spaces $\mathcal{M} = \mathcal{C}$ and $\mathcal{M} = B_\beta$ and for both types of fundamental algorithms $\alpha = L$ and $\alpha = G$ the computational problem $\{\mathrm{H}, \mathcal{B}_\beta, \mathcal{M}, \Lambda\}$ has the same solvability complexity index SCI $= 2$. This follows immediately from (5) and (6) because

$$\begin{aligned} 2 = \mathrm{SCI}_\mathrm{H}\left(\mathcal{B}_\beta\right)_L &\geq \mathrm{SCI}_\mathrm{H}\left(\mathcal{C}\right)_L \geq \mathrm{SCI}_\mathrm{H}\left(\mathcal{C}\right)_G = 2 \\ \text{and} \quad 2 = \mathrm{SCI}_\mathrm{H}\left(\mathcal{B}_\beta\right)_L &\geq \mathrm{SCI}_\mathrm{H}\left(\mathcal{B}_\beta\right)_G \geq \mathrm{SCI}_\mathrm{H}\left(\mathcal{C}\right)_G = 2\,. \end{aligned}$$

So not only $\mathrm{SCI}_\mathrm{H}\left(\mathcal{B}_\beta\right)_L = \mathrm{SCI}_\mathrm{H}\left(\mathcal{C}\right)_G = 2$, as claimed by the theorem, but also $\mathrm{SCI}_\mathrm{H}\left(\mathcal{C}\right)_L = \mathrm{SCI}_\mathrm{H}\left(\mathcal{B}_\beta\right)_G = 2$.

Proof: We show first that $\mathrm{SCI}\left(\mathrm{H}, \mathcal{B}_\beta, \mathcal{C}, \Lambda\right)_G \geq 2$. To this end, we note that $\mathrm{SCI}\left(\mathrm{H}, \mathcal{B}_\beta, \mathcal{C}, \Lambda\right)_G \geq 1$. Otherwise, it would be possible to write every $f \in \mathcal{B}_\beta$ as a finite sampling series. Since $\mathcal{B}_\beta$ is an infinite-dimensional space, this is impossible. Assume now that $\mathrm{SCI}\left(\mathrm{H}, \mathcal{B}_\beta, \mathcal{C}, \Lambda\right)_G = 1$. Then there exists a sequence $\{\Gamma_{n_1}\}_{n_1 \in \mathbb{N}}$ of functions satisfying the conditions for the fundamental algorithms given in Def. 2 and such that

$$\lim_{n_1 \to \infty} \|\mathrm{H}f - \Gamma_{n_1}(f)\|_\mathcal{C} = 0 \qquad \text{for all } f \in \mathcal{B}_\beta\,.$$

Nevertheless, this would contradict Theorem 2 and so the SCI of our problem has to be larger than one, i.e.

$$\mathrm{SCI}\left(\mathrm{H}, \mathcal{B}_\beta, \mathcal{C}, \Lambda\right)_G \geq 2\,. \tag{7}$$

Next, by constructing a tower of algorithms of height 2 and of type L, we are going to show that

$$\mathrm{SCI}\left(\mathrm{H}, \mathcal{B}_\beta, \mathcal{B}_\beta, \Lambda\right)_L \leq 2\,. \tag{8}$$

To this end, let $n_1, n_2 \in \mathbb{N}$ be fixed. For $f \in \mathcal{B}_\beta$, we consider

$$c_k(f, n_1) = \frac{1}{n_1} \sum_{l=0}^{n_1-1} f\left(l \frac{2\pi}{n_1}\right) \mathrm{e}^{-\mathrm{i}\frac{2\pi}{n_1} lk} , \qquad \text{for all } |k| \leq n_2 ,$$

and notice that $c_k(f, n_1)$ is the approximation of the kth Fourier coefficient (1) of f by a Riemann sum based on n_1 equally spaced sampling points. Therefore, and since f is continuous, we have

$$\lim_{n_1\to\infty} c_k(f, n_1) = c_k(f) \qquad \text{for all } |k| \leq n_2 . \tag{9}$$

Now we define the operators on the lowest level of our tower by

$$(\Gamma_{n_2,n_1} f)(t) = -\mathrm{i} \sum_{k=-n_2}^{n_2} \operatorname{sgn}(k)\, c_k(f, n_1)\, \mathrm{e}^{\mathrm{i}kt} , \qquad t \in \mathbb{T} ,$$

and the operators on the upper level by

$$(\Gamma_{n_2} f)(t) = -\mathrm{i} \sum_{k=-n_2}^{n_2} \operatorname{sgn}(k)\, c_k(f)\, \mathrm{e}^{\mathrm{i}kt} , \qquad t \in \mathbb{T} . \tag{10}$$

Clearly, all of these operators are of type L and it remains to verify that these operators satisfy the conditions of Def. 3. Indeed, since

$$\begin{aligned}(\Gamma_{n_2} f)(t) &- (\Gamma_{n_2,n_1} f)(t) \\ &= -\mathrm{i} \sum_{k=-n_2}^{n_2} \operatorname{sgn}(k) \left[c_k(f) - c_k(f, n_1)\right] \mathrm{e}^{\mathrm{i}kt}\end{aligned}$$

the triangle inequality and (9) easily yield

$$\lim_{n_1\to\infty} \|\Gamma_{n_2} f - \Gamma_{n_2,n_1} f\|_{\mathcal{B}_\beta} = 0 \qquad \text{for all } f \in \mathcal{B}_\beta ,$$

and for every finite $n_2 \in \mathbb{N}$.

It remains to verify that the upper level $\Gamma_{n_2} f$ of the tower converges to $\mathrm{H}f$ in the norm of $\mathcal{B}_\beta$. So by the definition of the norm in $\mathcal{B}_\beta$, we have to investigate the three terms

$$\big\|\widetilde{f} - \Gamma_{n_2} f\big\|_{\mathcal{C}} , \ \big\|\mathrm{H}\big[\widetilde{f} - \Gamma_{n_2} f\big]\big\|_{\mathcal{C}} , \ \big\|\widetilde{f} - \Gamma_{n_2} f\big\|_{\beta} . \tag{11}$$

To this end, we notice that (10) is just the partial Fourier series of the conjugate function $\widetilde{f}$, i.e.

$$\big(\Gamma_{n_2} f\big)(t) = \sum_{k=-n_2}^{n_2} c_k\big(\widetilde{f}\big)\, \mathrm{e}^{\mathrm{i}kt} = \big(\mathrm{P}_{n_2} \widetilde{f}\big)(t) .$$

Since $f \in \mathcal{B}_\beta$, Lemma 1 implies immediately that the first term in (11) converges to zero as $n_2 \to \infty$, i.e.

$$\lim_{n_2\to\infty} \big\|\mathrm{H}f - \Gamma_{n_2} f\big\|_{\mathcal{C}} = \lim_{n_2\to\infty} \big\|\widetilde{f} - \mathrm{P}_{n_2}\widetilde{f}\big\|_{\mathcal{C}} = 0 . \tag{12}$$

For the second term in (11), we note first that

$$\begin{aligned}\mathrm{H}\big(\widetilde{f} - \Gamma_{n_2} f\big) &= \mathrm{H}\big(\widetilde{f} - \mathrm{P}_{n_2}\widetilde{f}\big) = \mathrm{H}\big(\mathrm{H}f - \mathrm{P}_{n_2}\mathrm{H}f\big) \\ &= \mathrm{H}\big(\mathrm{H}f - \mathrm{H}\mathrm{P}_{n_2} f\big) = \mathrm{H}\mathrm{H}\big(f - \mathrm{P}_{n_2} f\big) = -\big(f - \mathrm{P}_{n_2} f\big) .\end{aligned}$$

Then Lemma 1 shows that the right hand side converges uniformly to zero as $n_2 \to \infty$, i.e.

$$\lim_{n_2\to\infty} \big\|\mathrm{H}\big(\widetilde{f} - \Gamma_{n_2} f\big)\big\|_{\mathcal{C}} = \lim_{n_2\to\infty} \big\|f - \mathrm{P}_{n_2} f\big\|_{\mathcal{C}} = 0 . \tag{13}$$

Finally, for the last term in (11), we have

$$\begin{aligned}\big\|\widetilde{f} - \Gamma_{n_2} f\big\|_{\beta} &= \big\|\widetilde{f} - \mathrm{P}_{n_2}\widetilde{f}\big\|_{\beta} \\ &= \Big(\sum_{|k|>n_2} |k|\ (1 + \log|k|)^{\beta}\ |c_k(f)|^2\Big)^{1/2}\end{aligned}$$

showing that $\lim_{n_2\to\infty} \big\|\widetilde{f} - \Gamma_{n_2} f\big\|_{\beta} = 0$. Combining this observation with (12) and (13), one obtains the desired result, namely

$$\lim_{n_2\to\infty} \big\|\widetilde{f} - \Gamma_{n_2} f\big\|_{\mathcal{B}_\beta} = \lim_{n_2\to\infty} \big\|\mathrm{H}f - \Gamma_{n_2} f\big\|_{\mathcal{B}_\beta} = 0 .$$

So we verified that the pair of sequences $\{\Gamma_{n_2,n_1}\}$ and $\{\Gamma_{n_2}\}$ is a tower of algorithms of height 2 and type L, i.e. we verified (8).

Combining (7), (8) with observations (5), we get finally

$$2 \leq \mathrm{SCI}_{\mathrm{H}}\left(\mathcal{C}\right)_G \leq \mathrm{SCI}_{\mathrm{H}}\left(\mathcal{B}_\beta\right)_G \leq \mathrm{SCI}_{\mathrm{H}}\left(\mathcal{B}_\beta\right)_L \leq 2 ,$$

which proves the statement of the theorem. ■

Notice that the tower of algorithms for sampling based Hilbert transform approximations, as given in the previous proof, is very simple. The first layer of the tower determines basically the Fourier coefficients $c_k(f)$ by an approximation of the Fourier integral by a Riemann sum (first limit). Then the second layer determines the Hilbert transform via the conjugate Fourier series (second limit) based on the Fourier coefficients of f. It is an interesting result that for approximations with 2 limits such a simple algorithm already works. For algorithms with only 1 limit, on the other hand, even arbitrary (non-linear) sampling operators yield no convergent method (cf. Theorem 2).

VI. Discussion and Conclusions

Previous works already showed that there is no sampling based algorithm which is able to determine the Hilbert transform for all functions in $\mathcal{B}_\beta$ with $\beta \in [0, 1]$ as the limit of a one-parameter approximation process. Applying the new framework of the solvability complexity index (SCI) and towers of algorithms, this paper showed that there exist sampling based towers of algorithms of height 2 for the calculation of the Hilbert transform. So the SCI for sampling based Hilbert transform approximations is priestly 2, i.e. the Hilbert transform can be determined by an approximation procedure which involves two limits. It is an interesting and surprising observation that the SCI is the same for towers of algorithms of type G as well as for type L towers. So in the scale of the SCI, general algorithms have no advantage compared to linear algorithms.

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