Generalized Sampling on Graphs With A Subspace Prior

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***Abstract*—We consider a framework for generalized sampling of graph signals that extends sampling results in shift-invariant (SI) spaces to the graph setting. We assume that the input signal lies in a periodic graph spectrum subspace, which generalizes the standard SI assumption to graph signals. Sampling is performed in the graph frequency domain by an arbitrary graph filter. We show that under a mild condition on the sampling filter, perfect recovery is possible using a correction filter that can be represented as a spectral graph filter whose response depends on the prior subspace spectrum and on the sampling filter. This filter parallels the correction filter in SI sampling in standard signal processing. Since the input space and the sampling filter are almost arbitrary, our framework allows perfect recovery of many classes of input signals from a variety of different sampling patterns using a simple correction filter. For example, our method enables perfect recovery of non-bandlimited graph signals from their bandlimited measurements.**

I. Introduction

Sampling theory for graph signal processing has been recently studied with the goal of building a parallel of sampling results in standard signal processing [1]–[4]. Conventional sampling theory allows sampling and recovery of signals in arbitrary subspaces using an almost arbitrary sampling kernel [5]. These general results are particularly useful in the shift-invariant (SI) setting where sampling and recovery reduce to simple filtering operations [6].

To date, most works on sampling graph signals consider recovery of discrete graph signals from their sampled version [1]–[4], [7], [8]. Current approaches to sampling of graph signals generally rely on vertex sampling. Due to the irregular nature of the graph, the vertex set used for sampling is generally distributed nonuniformly. Its counterpart in the time domain is general nonuniform sampling of an input. An extension of generalized sampling to graphs was recently proposed in [4]. This framework allows for arbitrary subspaces and arbitrary sampling operators and offers a recovery method that is in general given by operator inversion.

In this paper, our goal is to build a graph sampling framework which parallels SI sampling for time domain signals. In SI sampling, the input subspace has a particular SI structure. Sampling is modeled by uniformly sampling the output of the signal convolved with an arbitrary sampling filter. Under a mild condition on the sampling filter, recovery is obtained by a correction filter which has an explicit closed-form frequency response. Here we show how one can extend these ideas to graphs by defining an appropriate input space of graph signals and sampling in the graph frequency domain. The recovery is then given by a spectral graph filter whose response depends on the prior subspace spectrum and the sampling filter. More specifically, our framework relies on the following ingredients:

- We assume an input signal space called *periodic graph spectrum* (PGS) subspace as a counterpart of the SI subspace; and
- Sampling in the graph frequency domain [9] as a counterpart of uniform sampling at the output of a filter.

Our approach reduces to the standard SI results in the case of a graph representing the conventional time axis, and allows recovery over graphs without having to invert general operators. We also allow perfect recovery of non-bandlimited graph signals, in contrast to many prior sampling results on graphs.

We begin by defining a new class of subspaces, the PGS subspace, which maintains the frequency structure of SI subspaces in the graph frequency domain. The PGS subspace depends on the given underlying graph and its associated graph Fourier basis. We then define sampling in the spectral domain by extending the notion of filtering and uniform subsampling to arbitrary graphs. In particular, the combined affect of the later operation in standard Fourier analysis is multiplying the input spectrum by the sampling filter's frequency response and then aliasing in the frequency domain. We define graph sampling by these operations extended to the graph Fourier basis. With these definitions we show that a graph signal in any PGS space can be recovered from its arbitrary samples using a simple graph spectral filter.

This paper is organized as follows. Section II introduces generalized sampling in SI spaces and sampling in the graph frequency domain. The proposed framework of generalized graph sampling is presented in Section III. Section IV describes relationships between ours and some existing works. A numerical experiment for recovery of non-bandlimited graph signals is shown in Section V. Finally, Section VI concludes the paper.

II. SI and Graph Sampling

A. Generalized Sampling in Shift-Invariant Spaces

Our goal is to propose a generalized graph signal sampling method that parallels sampling in SI spaces in standard signal processing [5], [6].

Sampling and recovery in SI spaces is illustrated in Fig. 1. The original (continuous-time) signal $x(t)$ is filtered by a *sampling filter* $s(t)$ and then uniformly sampled with sampling period T to generate the sampled (discrete) coefficients $c[n]$. This sampling operation can be formulated as

$$c[n] = \langle s(t-nT), x(t)\rangle = x(t) * s(-t)|_{t=nT}\,, \tag{1}$$

where we assume $s(t)$ satisfies the Riesz condition. In bandlimited sampling, $s(-t) = \mathrm{sinc}(t/T)$, where $\mathrm{sinc}(t) = \sin(\pi t)/(\pi t)$. However, $s(t)$ can be arbitrary in the generalized sampling framework.

The continuous-time Fourier transform (CTFT) of the samples $c[n]$, denoted $C(\omega)$, can be represented as [10]:

$$C(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} S^*\left(\frac{\omega - 2\pi k}{T}\right) X\left(\frac{\omega - 2\pi k}{T}\right). \tag{2}$$

Thus, we can view sampling in the Fourier domain as multiplying the input spectrum by the filter's frequency response, and then aliasing the result with uniform intervals that depend on the sampling period. In our definition of graph sampling, we rely on this frequency-domain interpretation.

To recover $x(t)$ from the samples, we assume that it is known to lie in a SI subspace, namely,

$$x(t) = \sum_{n\in\mathbb{Z}} d[n]a(t-nT), \tag{3}$$

for some sequence $d[n]$ where $a(t)$ is a (known) real generator satisfying the Riesz condition. In the Fourier domain, this prior can be expressed as

$$X(\omega) = D(e^{j\omega T})A(\omega), \tag{4}$$

where $A(\omega)$ is the CTFT of $a(t)$ and $D(e^{j\omega T})$ is the discrete-time Fourier transform (DTFT) of the sequence $d[n]$, and is $2\pi/T$ periodic.

When $x(t)$ lies in a SI subspace, it can be recovered from its samples in Fig. 1 using a correction filter $h[n]$. The corrected discrete signal $d[n]$ is interpolated by a continuous *interpolation filter* $w(t)$ to yield the reconstructed signal $\tilde{x}(t)$. By choosing $w(t) = a(t)$ and

$$H(\omega) = \frac{1}{\frac{1}{T}\sum_{k=-\infty}^{\infty} S^*\left(\frac{\omega-2\pi k}{T}\right) A\left(\frac{\omega-2\pi k}{T}\right)}, \tag{5}$$

we have perfect recovery $\tilde{x}(t) = x(t)$. Note that this approach does not require the input signal to be bandlimited.

B. Spectral Graph Theory

A graph $\mathcal{G}$ is represented as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ and $\mathcal{E}$ denote sets of vertices and edges, respectively. The number of vertices is given as $N = |\mathcal{V}|$ unless otherwise specified. We define an adjacency matrix $\mathbf{A}$ with elements a_{mn} that represents the weight of the edge between the mth and nth vertices; $a_{mn} = 0$ for unconnected vertices. The degree matrix $\mathbf{D}$ is a diagonal matrix, with mth diagonal element $[\mathbf{D}]_{mm} = \sum_n a_{mn}$.

Graph signal processing uses different variation operators [11], [12] depending on the application and assumed signal and/or network models. Here, for concreteness, we use the graph Laplacian $\mathbf{L} := \mathbf{D} - \mathbf{A}$ or its symmetrically normalized version $\underline{\mathbf{L}} := \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$. The extension to other variation operators (e.g., adjacency matrix) is possible with a slight modification for properly ordering its eigenvalues as long as the graph is undirected without self-loops. Since $\mathbf{L}$ is a real symmetric matrix, it always has an eigendecomposition $\mathbf{L} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^*$, where $\mathbf{U} = [\boldsymbol{u}_0, \dots, \boldsymbol{u}_{N-1}]$ is a unitary matrix containing the eigenvectors $\boldsymbol{u}_i$, and $\mathbf{\Lambda} = \mathrm{diag}(\lambda_0, \lambda_1, \dots, \lambda_{N-1})$ consists of the eigenvalues λ_i. We refer to λ_i as the *graph frequency*.

A graph signal $\boldsymbol{x} \in \mathbb{C}^N$ contains elements $x[n]$ that represent the signal value at the nth vertex. The graph Fourier transform (GFT) is defined as $\hat{x}[i] = \langle \boldsymbol{u}_i, \boldsymbol{x}\rangle = \sum_{n=0}^{N-1} u_i^*[n]x[n]$. Filtering in the graph Fourier domain is defined as generalized convolution [13]: $\boldsymbol{x}_{\text{out}} := \mathbf{U}G(\mathbf{\Lambda})\mathbf{U}^*\boldsymbol{x}$, where the filter response in the graph frequency domain is given by $G(\mathbf{\Lambda}) := \mathrm{diag}(G(\lambda_0), G(\lambda_1), \dots)$, with $G(\lambda_i) \in \mathbb{R}$.

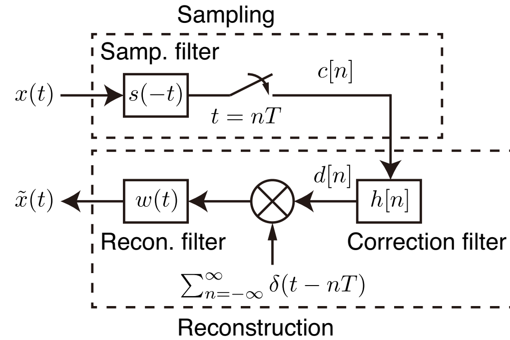

Fig. 1. Generalized sampling framework for SI spaces. The signals $x(t)$ and $\widehat{x}(t)$ are the original and reconstructed signals. The sequences $c[n]$ and $d[n]$ are the discrete-time samples and the corrected samples.

C. Sampling in the Graph Frequency Domain

To define sampling over a graph, we extend sampling in SI spaces expressed by (2) to the graph frequency domain [9]. In particular, the graph Fourier transformed input $\hat{\boldsymbol{x}}$ is first multiplied by a graph frequency filter $S(\mathbf{\Lambda})$; the product is then aliased with period K. This leads to the following definition.

Definition 1 (Sampling of graph signals in the graph frequency domain)**.** *Let $\hat{\boldsymbol{x}} \in \mathbb{C}^N$ be the original signal in the graph frequency domain, i.e., $\hat{\boldsymbol{x}} = \mathbf{U}^*\boldsymbol{x}$, and let $S(\mathbf{\Lambda})$ be an arbitrary sampling filter in the graph frequency domain. For any sampling ratio $M \in \mathbb{Z}$, the sampled graph signal in the graph frequency domain*[1] *is given by $\hat{\boldsymbol{c}} \in \mathbb{C}^K$, where $K = N/M$, and*

$$\hat{c}(\lambda_i) = \sum_{p=0}^{M-1} S(\lambda_{i+pK})\,\hat{x}(\lambda_{i+pK})\,. \tag{6}$$

In matrix form, the downsampled graph signal can be represented as $\hat{\boldsymbol{c}} = \mathbf{D}_{\text{samp}}S(\mathbf{\Lambda})\hat{\boldsymbol{x}}$ with $\mathbf{D}_{\text{samp}} = \begin{bmatrix}\mathbf{I}_K & \mathbf{I}_K & \dots\end{bmatrix}$.

[1] M is assumed to be a divisor of N for simplicity.

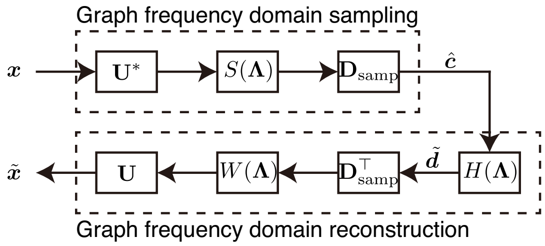

Fig. 2. Generalized sampling framework in the graph frequency domain.

When the eigenvector matrix $\mathbf{U}^*$ is the DFT matrix, this definition coincides with DFT-domain sampling.

III. Graph Signal Recovery with a Subspace Prior

To consider subspace sampling for graph signals, we first introduce a generalization of (4) to graphs.

A. Periodic Graph Spectrum Subspace

We assume that the graph signal has a periodic GFT spectrum that parallels (4). In other words, $\hat{x}[i] = \hat{d}(\lambda_{i \bmod K})A(\lambda_i)$ where $A(\lambda_i)$ is the graph frequency domain response of a generator and $\hat{d}(\lambda_k)$ for $k = 0, \ldots, K-1$ are the expansion coefficients which are repeated periodically. Formally, the PGS subspace is defined as follows:

Definition 2. *A PGS subspace of a given graph $\mathcal{G}$ is a space of graph signals that can be expressed as*

$$\mathcal{W} = \left\{ x[n] \,\middle|\, x[n] = \sum_{i=0}^{N-1} \hat{d}(\lambda_{i \bmod K})A(\lambda_i)u_i[n] \right\} \quad (7)$$

where $A(\lambda_i)$ is an arbitrary graph frequency domain response and $\hat{d}(\lambda_k)$ for $k = 0, \ldots, K-1$ are expansion coefficients.

A PGS signal can be represented in matrix form as:

$$\boldsymbol{x} := \mathbf{A}\hat{\boldsymbol{d}} = \mathbf{U}A(\mathbf{\Lambda})\mathbf{D}_{\text{samp}}^{\top}\hat{\boldsymbol{d}}. \quad (8)$$

B. Sampling and Reconstruction Framework

With our definitions of PGS and GFT sampling, sampling and reconstruction of signals over graphs can be represented as in Fig. 2. This framework parallels that of SI sampling described in Fig. 1, with all operations now performed in the graph frequency domain. The important aspect is that the correction filter has a diagonal graph spectral response, and therefore can be implemented simply without matrix inversion.

We assume the original signal $\boldsymbol{x} \in \mathbb{C}^N$ lies in a PGS subspace characterized by $\mathbf{A}$. To sample it, we transform the input to the GFT resulting in $\hat{\boldsymbol{x}} = \mathbf{U}^*\boldsymbol{x}$. The output is then filtered by the sampling filter $S(\mathbf{\Lambda})$. The filtered signal is downsampled to yield the sampled signal $\hat{\boldsymbol{c}} := \mathbf{D}_{\text{samp}}S(\mathbf{\Lambda})\hat{\boldsymbol{x}}$. In the reconstruction step, $\hat{\boldsymbol{c}}$ is filtered by the correction filter $\mathbf{H} = H(\mathbf{\Lambda})$. Finally, $\tilde{\boldsymbol{d}} = H(\mathbf{\Lambda})\hat{\boldsymbol{c}}$ is upsampled to the original dimension by $\mathbf{D}_{\text{samp}}^{\top}$, and the reconstruction filter $A(\mathbf{\Lambda})$ is applied to the upsampled signal. After performing the inverse GFT, we obtain the recovered signal $\tilde{\boldsymbol{x}}$. We now show how to choose the correction filter $\mathbf{H}$ to enable perfect recovery.

Denote by $\mathbf{S}$ the sampling matrix. From Definition 1,

$$\mathbf{S}^* = \mathbf{D}_{\text{samp}}\text{diag}(S(\mathbf{\Lambda}_0), S(\mathbf{\Lambda}_1), \ldots, S(\mathbf{\Lambda}_{M-1}))\mathbf{U}^*, \quad (9)$$

where $S(\mathbf{\Lambda}_\ell) := \text{diag}(S(\lambda_{K\ell}), S(\lambda_{K\ell+1}), \ldots, S(\lambda_{K(\ell+1)-1}))$. To choose $\mathbf{H}$ we consider the following minimization problem:

$$\min_{\mathbf{H}} \|\mathbf{H}\mathbf{S}^*\mathbf{A}\hat{\boldsymbol{d}} - \hat{\boldsymbol{d}}\|_2^2, \quad (10)$$

where the left-hand expression is the reconstruction of $\hat{\boldsymbol{d}}$.

Let us denote the subspaces spanned by $\mathbf{A}$ and $\mathbf{S}$ as $\mathcal{A}$ and $\mathcal{S}$, respectively. Suppose that $\mathcal{A}$ and $\mathcal{S}^\perp$ intersect only at the origin and together span $\mathbb{C}^N$. In this case, (10) has a unique solution that guarantees perfect recovery [5]. By solving (10) for $\mathbf{H}$, we obtain

$$\mathbf{H} = (\mathbf{S}^*\mathbf{A})^{-1} \quad (11)$$

where the inverse always exists with the above condition on $\mathcal{A}$ and $\mathcal{S}^\perp$. The recovery is then given by

$$\begin{aligned} \tilde{\boldsymbol{x}} &= \mathbf{A}(\mathbf{S}^*\mathbf{A})^{-1}\mathbf{S}^*\boldsymbol{x} \\ &= \mathbf{A}(\mathbf{S}^*\mathbf{A})^{-1}\mathbf{S}^*\mathbf{A}\hat{\boldsymbol{d}} \\ &= \mathbf{A}\hat{\boldsymbol{d}} = \boldsymbol{x}, \end{aligned} \quad (12)$$

where (8) is used for the calculation from the first to the second lines. Substituting (8) and (9) into (11), we have the explicit expression

$$H(\lambda_k) = \frac{1}{\sum_\ell S(\lambda_{k+K\ell})A(\lambda_{k+K\ell})}. \quad (13)$$

Note the similarity with (5).

As a special case, suppose that both the generator and sampling filters are bandlimiting filters $A(\lambda_i) = S(\lambda_i) = 1$ for $i \in [0, K-1]$ and 0 otherwise. Then $H(\lambda_i) = 1$ and no correction filter is needed. This is equivalent to the perfect recovery condition for bandlimited graph signals using frequency domain sampling [9].

IV. Relationship to Prior Works on Graph Sampling

In [4], a generalized sampling method for graph signal processing has been studied. Since the results did not assume any particular structure on the input signals and sampling filters, the recovery procedures where in general given by matrix inversions. Here we focus on a special case of [4] that extends SI sampling to the graph setting and enables explicit expressions for the recovery filter in the graph Fourier domain. Our solution represented in (12) allows for a broad choice of $S(\lambda)$ and $A(\lambda)$. In particular, $A(\lambda)$ is not restricted to be a bandlimiting operator. If we have $S(\mathbf{\Lambda}) = A(\mathbf{\Lambda}) = \text{diag}(\mathbf{I}_K, \mathbf{0})$, then our solution reduces to that of [4], which is also equivalent to sampling theory with graph frequency domain sampling studied in [9].

Many works in graph sampling theory like [1]–[3] assume that the graph signal is K-bandlimited. That means only K elements in $\hat{\boldsymbol{x}}$ are nonzero. Hereafter, we assume its first K elements are nonzero for simplicity. This assumption implies that the signal subspace is $\mathbf{A} = \mathbf{U}A_{\text{BL}}(\mathbf{\Lambda})\mathbf{D}_{\text{samp}}^{\top}$ where

$A_{\mathrm{BL}}(\boldsymbol{\Lambda}) := \mathrm{diag}(\mathbf{I}_K, \mathbf{0}_{N-K})$ is the binary response of the graph low-pass filter, that bandlimits the graph signal. In other words, they implicitly assume the graph signal lies in the PGS subspace without periodicity of the spectrum; it is removed by $A_{\mathrm{BL}}(\boldsymbol{\Lambda})$.

While their subspace is a special case of the PGS assumption, the sampling matrices are different. Let us define $\mathbf{I}_{\mathcal{T}} \in \{0,1\}^{K\times N}$ as a submatrix of the identity matrix $\mathbf{I}_N$ whose rows are specified by $\mathcal{T}$, i.e., vertex indices remaining after sampling. This can be regarded as a nonuniform subsampling matrix. The works of [1], [2] simply use $\mathbf{I}_{\mathcal{T}}$ as $\mathbf{S}^*$. The paper [3] utilizes aggregation sampling that defines sampling as observations gathered at a single vertex i. In this case, $\mathbf{S}^* = \mathbf{I}_{\mathcal{T}}\boldsymbol{\Psi}\mathrm{diag}(u_0^*(\lambda_i), u_1^*(\lambda_i), \dots)\mathbf{U}^*$, where $[\boldsymbol{\Psi}]_{k,\ell} = \lambda_\ell^k$. Such matrices do not in general have a corresponding sampling expression in the graph frequency domain as in (6). This leads to the requirement of matrix inversion even for recovering bandlimited graph signals though the signal lies in a PGS subspace.

V. Signal Recovery Experiments

In this section, we validate our generalized sampling theory via a toy example. The graph used is a random sensor graph with $N = 64$. We downsample the input signal by a factor of two so that $K = 32$. We consider the following functions:

- Generator function: $A(\lambda_i) = 1 - 2\lambda_i/\lambda_{\max}$
- Sampling function: $S(\lambda_i) = \begin{cases} 1 & i \le 32 \\ 0 & i > 32. \end{cases}$

The correction $H(\lambda_i)$ is designed from $A(\lambda_i)$ and $S(\lambda_i)$ by using (13). Each element in the expansion coefficients $\hat{\boldsymbol{d}}$ is a random variable drawn from $\mathcal{N}(1,1)$. The original signal $\boldsymbol{x}$ generated by $A(\lambda)$ is a full-band signal as shown in Fig. 3(a). We sample the signal with a bandlimiting function. For comparison, we also perform signal recovery using a recovery filter $S(\lambda_i)$ with no correction filter.

Fig. 3 shows the original and reconstructed signals and corresponding spectra. As can be seen, the input signal is perfectly recovered with machine precision, while the bandlimiting filter yields large error.

VI. Conclusions

In this paper, we presented a framework for generalized sampling of graph signals that extends SI sampling to the graph Fourier domain. Our approach is based on defining a PGS input space of graph signals, and defining sampling in the Fourier domain. This enables perfect recovery of a broad class of input signals which are not necessarily bandlimited using a simple filter in the graph frequency domain. In particular, we demonstrate via an example that perfect recovery of non-bandlimited graph signals is possible from bandlimited measurements.

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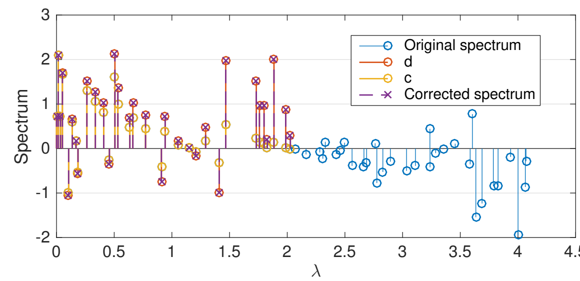
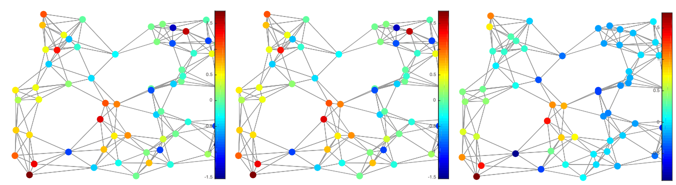


(a) Spectra



(b) Original (c) Reconstructed with the subspace prior (d) Reconstructed with the bandlimited filter

Fig. 3. Experimental results of signal recovery experiments. MSEs of signal recovery with the subspace prior and with the binary bandlimiting filters are 3.20×10^{-28} and 2.04×10^{-1}, respectively.

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