Computability of the Fourier Transform and ZFC

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***Abstract*—In this paper we study the Fourier transform and the possibility to determine the binary expansion of the values of the Fourier transform in the Zermelo–Fraenkel set theory with the axiom of choice included (ZFC). We construct a computable absolutely integrable bandlimited function with continuous Fourier transform such that ZFC (if arithmetically sound) cannot determine a single binary digit of the binary expansion of the Fourier transform at zero. This result implies that ZFC cannot determine for every precision goal a rational number that approximates the Fourier transform at zero. Further, we discuss connections to Turing computability.**

I. Introduction

The Fourier transform is of fundamental importance in signal processing and other disciplines such as physics and mathematics [1]–[6]. In order to be of practical use it is essential that there is a way to compute it. Although the Fourier transform is such an important operation there are little investigations about its computability, except in the discrete, i.e. finite setting [7]–[11].

We view the Fourier transform as a mapping and are interested in the computability and approximability of the output functions. Our goal is to find a function in the time-domain that is as simple as possible such that the Fourier transform of this function has still nice analytical properties but is hard to approximate. To this end we construct a bandlimited function with continuous Fourier transform, of which we even know where the maximum is attained, such that this maximum, or, in other words, the Fourier transform at a certain point, cannot be determined algorithmically. We even go one step further, by proving that the strongest of the usually employed axiomatic mathematical theories, Zermelo–Fraenkel set theory with the axiom of choice included (ZFC), is not sufficient to prove statements about the binary representation or the possibility to approximate the Fourier transform at this point by rational numbers. More specifically, we construct a Turing computable absolutely integrable bandlimited function f_* such that its Fourier transform $\hat{f}_*$ is continuous but ZFC (if arithmetically sound) cannot determine a single binary digit of the binary expansion of $\hat{f}_*(0)$.

In [12, p. 110, Th. 4] the computability of the Fourier transform was studied for certain $L^p(\mathbb{R})$ spaces, and type-2 computability was studied in [13]. An example of a computable continuous function that has a non-computable Fourier transform was presented in [14].

II. Notation

Let $\mathbb{N}$ denote the natural numbers. By C we denote the space of all continuous functions on $\mathbb{R}$, equipped with the norm $\|\cdot\|_\infty$. For $\Omega \subseteq \mathbb{R}$, let $L^p(\Omega)$, $1 \leq p < \infty$, be the space of all measurable, pth-power Lebesgue integrable functions on Ω, with the usual norm $\|\cdot\|_p$, and $L^\infty(\Omega)$ the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite. The Bernstein space $\mathcal{B}_\sigma^p$, $\sigma > 0$, $1 \leq p \leq \infty$, consists of all functions of exponential type at most σ, whose restriction to the real line is in $L^p(\mathbb{R})$ [15, p. 49]. The norm for $\mathcal{B}_\sigma^p$ is given by the L^p-norm on the real line. A function in $\mathcal{B}_\sigma^p$ is called bandlimited to σ. We have $\mathcal{B}_\sigma^p \subset \mathcal{B}_\sigma^r$ for all $1 \leq p \leq r \leq \infty$ [15, p. 49, Lemma 6.6] and $\|f\|_r \leq C_{p,r}\|f\|_p$ for all $f \in \mathcal{B}_\sigma^p$, where $C_{p,r}(\sigma)$ is a constant that depends on p,r, and σ.

III. Basic Concepts of Computability Theory

The theory of computability is a well-established field in computer sciences [12], [16]–[19]. We describe some of the key concepts in this section. For a more detailed treatment of the topic, see for example [12], [18]–[20].

Alan Turing introduced the concept of a computable real number in [16], [17]. A sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ is called a computable sequence if there exist recursive functions a, b, s from $\mathbb{N}$ to $\mathbb{N}$ such that $b(n) \neq 0$ for all $n \in \mathbb{N}$ and $r_n = (-1)^{s(n)} \frac{a(n)}{b(n)}$, $n \in \mathbb{N}$. A recursive function is a function, mapping natural numbers into natural numbers, that is built of simple computable functions and recursions [21]. Recursive functions are computable by a Turing machine. A real number x is said to be computable if there exists a computable sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that $|x - r_{\xi(n)}| < 2^{-n}$ for all $n \in \mathbb{N}$. By $\mathbb{R}_c$ we denote the set of computable real numbers and by $\mathbb{C}_c = \mathbb{R}_c + i\mathbb{R}_c$ the set of computable complex numbers. $\mathbb{R}_c$ is a field, i.e., finite sums, differences, products, and quotients of computable numbers are computable. Note that commonly used constants like e and π are computable. A non-computable real number was constructed in [22].

IV. Computable Bandlimited Functions

There are several—non equivalent—definitions of computable functions, most notably, Turing computable functions, Markov computable functions, and Banach–Mazur computable functions [20]. A functions that is computable with respect to any of the above definitions, has the property that it maps computable numbers into computable numbers. This property is therefore a necessary condition for computability. Usual functions like sin, sinc, log, and exp are Turing computable, and finite sums of Turing computable functions are Turing computable [12]. An example of a non-computable function was given in [23].

We call a function f elementary computable if there exists a natural number N and a sequence of computable numbers $\{\alpha_k\}_{k=-N}^{N}$ such that

$$f(t) = \sum_{k=-N}^{N} \alpha_k \frac{\sin(\pi(t-k))}{\pi(t-k)}. \qquad (1)$$

Note that every elementary computable function f is a finite sum of Turing computable functions and hence Turing computable. As a consequence, for every $t \in \mathbb{R}_c$ the number $f(t)$ is computable. Further, the sum of finitely many elementary computable functions is computable, as well as the product of an elementary computable functions with a computable number $\lambda \in \mathbb{C}_c$. Hence, the set of elementary computable functions is closed with respect to the operations addition and multiplication with a scalar. Further, for every elementary computable function f, the norm $\|f\|_{\mathcal{B}_\pi^p}$, $p \in [1,\infty) \cap \mathbb{R}_c$, is computable.

A function in $f \in \mathcal{B}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, is computable in $\mathcal{B}_\pi^p$ if there exists a computable sequence of elementary computable functions $\{f_n\}_{n\in\mathbb{N}}$ such that $\|f - f_n\|_{\mathcal{B}_\pi^p} \le 2^{-n}$ for all $n \in \mathbb{N}$. By $\mathcal{CB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, we denote the set of all functions that are computable in $\mathcal{B}_\pi^p$. Note that $\mathcal{CB}_\pi^p$ has a linear structure. We can approximate every function $f \in \mathcal{CB}_\pi^\infty$ by an elementary computable functions, where we have an "effective" control of the approximation error. Computability for the space $\mathcal{B}_{\pi,0}^\infty$ (signals in $\mathcal{B}_\pi^\infty$ that converge to zero) is defined analogously with some minor technical differences. In this case the $\mathcal{B}_\pi^p$-norm is replaced by the $\mathcal{B}_\pi^\infty$-norm.

V. Chaitin Function

Partial functions on $\mathbb{N}$ are functions that may not be defined for all $n \in \mathbb{N}$. Partial recursive functions are exactly those functions that can be algorithmically computed with a Turing machine. The domain $\mathrm{dom}(\psi)$ of a partial recursive function ψ is recursively enumerable, i.e., there exists a recursive function $\phi\colon \mathbb{N} \to \mathrm{dom}(\psi)$ such that $\phi[\mathbb{N}] = \mathrm{dom}(\psi)$ and ϕ is a one-to-one function. Note that ϕ is a total function that is defined on all of $\mathbb{N}$. For more details see [21].

In the following we will consider special partial recursive functions and use connections between the theory of recursive functions and algorithmic information theory. We follow the notation used in [24]. Let Σ^* denote the set of all finite sequences of 0's and 1's. We call $u \in \Sigma^*$ a bit string and denote its length by $|u|$. For two bit strings u and v, $u \frown v$ denotes the concatenation of u and v. We can define a total order $<_{\Sigma^*}$ for the set Σ^* by putting $u <_{\Sigma^*} v$ if

1) $|u| < |v|$, or
2) $|u| = |v|$ and u lexicographically precedes v.

This ordering $0 < 1 < 00 < 01 < 10 < 11 < 000 < \dots$ provides a numbering of Σ^*, and thus a bijection between $\mathbb{N}$ and Σ^*. Hence, any partial recursive function $\psi\colon \mathbb{N} \to \mathbb{N}$ can be interpreted as a mapping from Σ^* into Σ^*.

A bit string $u \subset \Sigma^*$ is a prefix of a bit string $v \subset \Sigma^*$ if $v = u \frown r$ for some $r \in \Sigma^*$. $A \subset \Sigma^*$ is called prefix-free code, if for arbitrary $u, v \in A$ with the property that u is a prefix of v, we have $u = v$. For a prefix-free code $A \subset \Sigma^*$ we have the Kraft–Chaitin inequality

$$\Omega_A := \sum_{u \in A} \frac{1}{2^{|u|}} \le 1. \qquad (2)$$

We call a partial recursive function $\psi\colon \Sigma^* \to \Sigma^*$ a Chaitin function if its domain $\mathrm{dom}(\psi)$ is a prefix-free code.

By ZFC we denote the Zermelo–Fraenkel set theory with the axiom of choice included. ZFC is the common and accepted foundation of mathematics. Almost all mathematical statements can be formulated in a way that provable statements can be derived from ZFC. We call ZFC arithmetically sound if any sentence of arithmetic which is a theorem of ZFC is true in the standard model of Peano arithmetic (PA).

Let $A \subset \Sigma^*$ be the domain of a Chaitin function, and let $\phi_A\colon \mathbb{N} \to A$ be the recursive enumeration of the elements of A, created by the total order $<_{\Sigma^*}$. According to (2) we have

$$\Omega_A = \sum_{N=1}^{\infty} \frac{1}{2^{|\phi_A(N)|}} \le 1. \qquad (3)$$

VI. Binary Expansions

A rational number $x \in (0,1)$ is called dyadic rational if we have $x = m/2^N$ for some $m, N \in \mathbb{N}$. Without loss of generality we can assume that m and 2^N are coprime. For every number $x \in (0,1)$ that is not dyadic rational we have the unique representation

$$x = \sum_{n=1}^{\infty} a_n(x) \frac{1}{2^n}, \qquad (4)$$

where $a_n(x) \in \{0,1\}$, $n \in \mathbb{N}$. We call (4) the binary expansion of x. If $x \in (0,1)$ is a dyadic rational then it has two distinct binary expansions. We always choose the one that ends in an infinite sequence of zeros. We call $a_n(x)$ the n-th binary digit of x. Further, we say that ZFC can determine the n-th binary digit of x if in ZFC we can prove the statement "The n-th binary digit of x is k" for either $k = 0$ or $k = 1$.

Solovay proved in [24] the following theorem.

Theorem 1 (Solovay)**.** *There exists a Chaitin function ψ_*, such that ZFC, if arithmetically sound, can determine no single binary digit of Ω_{A_*}, where $A_* = \mathrm{dom}(\phi_*)$.*

VII. Fourier Transform and ZFC

In this section we construct a continuous bandlimited function $f_* \in \mathcal{B}_{2\pi}^1$ such that f_* is computable as an element of $\mathcal{B}_{2\pi}^p$ for all $1 < p < \infty$, $p \in \mathbb{R}_c$, but ZFC, if arithmetically sound, cannot determine a single binary digit of $\hat{f}_*(0)$.

Theorem 2. *We construct a function $f_* \in \mathcal{B}_{2\pi}^1$ such that:*

1) *f_* is computable as an element of $\mathcal{B}_{2\pi}^p$ for all $1 < p < \infty$, $p \in \mathbb{R}_c$, and as an element of $\mathcal{B}_{2\pi}^\infty$,*
2) *f_* has a continuous Fourier transform $\hat{f}_*$,*
3) *$\hat{f}_*(\omega) \in \mathbb{C}_c$ for all $\omega \in \mathbb{R}_c \setminus \{0\}$,*
4) *ZFC, if arithmetically sound, cannot determine a single binary digit of the binary expansion of $\hat{f}_*(0)$.*

Due to space constraints, we cannot give the complete proof of Theorem 2. Instead, we only provide a sketch of the main steps. We need several Lemmas.

Lemma 1. *For all $N \in \mathbb{N}$ and all $\omega \in \mathbb{R}$ we have*

$$\left|\sum_{k=1}^{N} \frac{1}{k}\sin(k\omega)\right| \leq \pi. \tag{5}$$

Lemma 2. *For all $N \in \mathbb{N}$ and all $0 < \delta < \pi$ we have*

$$\left|\sum_{k=1}^{N} \frac{1}{k}\cos(k\omega)\right| \leq \log\left(\frac{1}{\delta}\right) + \frac{5}{2} + 2\pi \tag{6}$$

for all $\omega \in \mathbb{R}$ satisfying $|\omega - k| \geq \delta$ for all $k \in \mathbb{Z}$.

The proofs of Lemmas 1 and 2, without explicit constants, can be found for example in [25, pp. 182–191]. The exact values of the constants on the right hand side of (5) and (6) are not important, it only matters that they are computable.

For $N \in \mathbb{N}$, let

$$g_N(t) = \sum_{k=1}^{N} \frac{1}{k}\left(\frac{\sin(\pi(t-k))}{\pi(t-k)}\right)^2, \quad t \in \mathbb{R}.$$

Lemma 3. *Let $1 < p < \infty$. There exists a constant $C_1(p)$ such that for all $N \in \mathbb{N}$ we have $\|g_N\|_p \leq C_1(p)$. For $1 < p < \infty$, $p \in \mathbb{R}_c$, the constant $C_1(p)$ is computable, and g_N is computable in $\mathcal{B}^p_{2\pi}$. Further, we have $\hat{g}_N(0) \geq \log(N+1)$ for all $N \in \mathbb{N}$.*

Proof of Theorem 2. For $N \in \mathbb{N}$, let

$$h_N(t) = \frac{g_N(t)}{\hat{g}_N(0)}, \quad t \in \mathbb{R}.$$

Since g_N is computable in $\mathcal{B}^p_{2\pi}$, $p \in (1,\infty) \cap \mathbb{R}_c$, and $\hat{g}_N(0)$ is a computable number, it follows that h_N is computable in $\mathcal{B}^p_{2\pi}$. We further have

$$\|h_N\|_p = \frac{\|g_N\|_p}{\hat{g}_N(0)} \leq \frac{C_1(p)}{\log(N+1)}, \tag{7}$$

for $p \in (1,\infty)$, where we used Lemma 3. We have $\|h_N\|_1 = \hat{h}_N(0) = 1$. Further, h_N is continuous, because $h_N \in \mathcal{B}^1_{2\pi}$.

Let

$$f_*(t) = \sum_{N=1}^{\infty} \frac{1}{2^{|\phi_{A_*}(N)|}} h_N(t), \quad t \in \mathbb{R}, \tag{8}$$

where A_* is the set from Theorem 1. Since

$$\sum_{N=1}^{\infty} \left\|\frac{1}{2^{|\phi_{A_*}(N)|}} h_N\right\|_{\mathcal{B}^1_{2\pi}} = \sum_{N=1}^{\infty} \frac{1}{2^{|\phi_{A_*}(N)|}} \leq 1,$$

according to (3), we see that the series in (8) converges in the $\mathcal{B}^1_{2\pi}$-norm, and that $f_* \in \mathcal{B}^1_{2\pi}$. Note that $f_* \in \mathcal{B}^1_{2\pi}$ implies that $\hat{f}_*$ is continuous because $f_* \in L^1(\mathbb{R})$. Moreover, for $p \in (1,\infty)$ and $M \in \mathbb{N}$, we have

$$\begin{aligned}\left\|f_* - \sum_{N=1}^{M} \frac{1}{2^{|\phi_{A_*}(N)|}} h_N\right\|_{\mathcal{B}^p_{2\pi}} &\leq \sum_{N=M+1}^{\infty} \frac{1}{2^{|\phi_{A_*}(N)|}} \|h_N\|_{\mathcal{B}^p_{2\pi}} \\ &\leq \sum_{N=M+1}^{\infty} \frac{C_1(p)}{2^{|\phi_{A_*}(N)|}\log(N+1)} \leq \frac{C_1(p)}{\log(M+2)},\end{aligned}$$

where we used (7) and (3). This shows that, for $p \in (1,\infty) \cap \mathbb{R}_c$, the computable sequence $\{\sum_{N=1}^{M} h_N / 2^{|\phi_{A_*}(N)|}\}_{M=1}^{\infty}$ converges effectively in the $\mathcal{B}^p_{2\pi}$-norm to f_*. Hence, f_* is computable in $\mathcal{B}^p_{2\pi}$ for all $p \in (1,\infty) \cap \mathbb{R}_c$. f_* is also computable in $\mathcal{B}^\infty_{2\pi}$ because we have $\|f\|_{\mathcal{B}^\infty_{2\pi}} \leq (1+2\pi)\|f\|_{\mathcal{B}^p_{2\pi}}$ for all $f \in \mathcal{B}^p_{2\pi}$, according to Nikol'skii's inequality [15, p. 49]. Since $f_* \in \mathcal{B}^1_{2\pi}$, we can apply Lebesgue's dominated convergence theorem, which gives $\hat{f}_*(\omega) = \sum_{N=1}^{\infty} \hat{h}_N(\omega)/2^{|\phi_{A_*}(N)|}$. Hence, we see that

$$\hat{f}_*(0) = \sum_{N=1}^{\infty} \frac{1}{2^{|\phi_{A_*}(N)|}} = \Omega_{A_*},$$

which implies that no single binary digit of the binary expansion of $\hat{f}_*(0) \notin \mathbb{R}_c$ can be computed according to Theorem 1.

It remains to prove item 3). Let $\omega \in (-2\pi, 2\pi) \setminus \{0\}$ be arbitrary but fixed and $\delta = \min\{|\omega|, 2\pi - \omega, 2\pi + \omega\}$. Then a short calculation shows that

$$\begin{aligned}|\hat{h}_N(\omega)| &= \frac{\hat{q}(\omega)}{\hat{g}_N(0)}\left|\sum_{k=1}^{N}\frac{1}{k}\cos(k\omega) - i\sum_{k=1}^{N}\frac{1}{k}\sin(k\omega)\right| \\ &\leq \frac{1}{\hat{g}_N(0)}\underbrace{\left(\log\left(\frac{1}{\delta}\right) + \frac{5}{2} + 2\pi + \pi\right)}_{=:C_2(\delta)} \leq \frac{C_2(\delta)}{\log(N+1)},\end{aligned}$$

where we used Lemmas 1 and 2 in the first inequality. The function $\hat{q}(\omega)$ is given by $\hat{q}(\omega) = 1 - |\omega|/(2\pi)$ for $|\omega| \leq 2\pi$ and by $\hat{q}(\omega) = 0$ otherwise. It follows that

$$\begin{aligned}\left|\hat{f}_*(\omega) - \sum_{N=1}^{M} \frac{1}{2^{|\phi_{A_*}(N)|}}\hat{h}_N(\omega)\right| &\leq \sum_{N=M+1}^{\infty} \frac{1}{2^{|\phi_{A_*}(N)|}}|\hat{h}_N(\omega)| \\ &\leq \frac{C_2(\delta)}{\log(M+2)} \sum_{N=M+1}^{\infty} \frac{1}{2^{|\phi_{A_*}(N)|}} \leq \frac{C_2(\delta)}{\log(M+2)}.\end{aligned}$$

For $\omega \in (-2\pi, 2\pi) \cap \mathbb{R}_c \setminus \{0\}$, $C_2(\delta)$ is computable and we see that the sequence $\{\sum_{N=1}^{M} \hat{h}_N(\omega)/2^{\phi_{A_*}(N)}\}_{M=1}^{\infty}$ of computable numbers converges effectively to $\hat{f}_*(\omega)$. This shows that $\hat{f}_*(\omega)$ is computable for all $\omega \in (-2\pi, 2\pi) \cap \mathbb{R}_c \setminus \{0\}$. Since $f_* \in \mathcal{B}^1_{2\pi}$ and f_* is continuous, we have $\hat{f}_*(\omega) = 0$ for all $|\omega| \geq 2\pi$. Hence, it follows that $\hat{f}_*(\omega)$ is computable for all $\omega \in \mathbb{R}_c \setminus \{0\}$. □

It immediately follows that ZFC, if arithmetically sound, cannot determine a single binary digit of the norm $\|\hat{f}_*\|_\infty$.

Corollary 1. *Let f_* be the same function as in Theorem 2. Then ZFC, if arithmetically sound, cannot determine a single binary digit of the binary expansion of $\|\hat{f}_*\|_\infty$.*

Proof. For $\omega \neq 0$ we have

$$|\hat{f}_*(\omega)| \leq \sum_{N=1}^{\infty} \frac{|\hat{g}_N(\omega)|}{2^{|\phi_{A_*}(N)|}\hat{g}_N(0)} \leq \sum_{N=1}^{\infty} \frac{1}{2^{|\phi_{A_*}(N)|}} = \hat{f}_*(0),$$

which shows that $\hat{f}_*(0)$ is the maximum of the function $|\hat{f}_*|$. Hence, according to Theorem 2, ZFC, if arithmetically sound, cannot determine a single binary digit of the binary expansion of $\|\hat{f}_*\|_\infty = \max_{\omega \in \mathbb{R}}|\hat{f}_*(\omega)| = \hat{f}_*(0)$. □

The result of Corollary 1 is surprising because we have $\hat{f}_*(\omega) \in \mathbb{C}_c$ for all $\omega \in \mathbb{R}_c \setminus \{0\}$ and $\lim_{\omega \to 0} \hat{f}_*(\omega) = \hat{f}_*(0) = \|\hat{f}_*\|_\infty$.

VIII. Turing Computability

Next, we briefly discuss the consequences of Theorem 2 for the Turing computability of the Fourier transform $\hat{f}_*$. In the proof of Theorem 2, a function $f_* \in \mathcal{B}^1_{2\pi}$ was constructed such that

$$\hat{f}_*(0) = \Omega_{A_*} = \sum_{N=1}^{\infty} \frac{1}{2^{|\phi_{A_*}(N)|}},$$

where A_* is the domain of the Chaitin function ψ_* from Theorem 1. The partial sums $x_l = \sum_{N=1}^{l} 2^{-|\phi_{A_*}(N)|}$ define a monotonically increasing sequence $\{x_l\}_{l\in\mathbb{N}}$ of dyadic rational numbers. This sequence is completely described by PA. Further, the Kraft–Chaitin inequality gives that $x_l \leq 1$ for all $l \in \mathbb{N}$. Hence, it follows from ZFC that the limit $\Omega_{A_*} = \lim_{l\to\infty} x_l$ exists and is unique. Ω_{A_*} is a transcendental number and hence not dyadic rational [24], [26]. Thus, we have $\alpha = |1/2 - \Omega_{A_*}| > 0$. However, ZFC, if arithmetically sound, cannot determine whether $1/2 - \Omega_{A_*} = \alpha$ or $1/2 - \Omega_{A_*} = -\alpha$, or, equivalently, whether $\Omega_{A_*} \in (0, 1/2)$ or $\Omega_{A_*} \in (1/2, 1)$.

The next theorem is a negative statement about the approximability of Ω_{A_*} by rational numbers in ZFC.

Theorem 3. *Let Ω_{A_*} be the number that was constructed in the proof of Theorem 2. There exists a natural number M_0 such that ZFC, if arithmetically sound, cannot prove the statement $|\Omega_{A_*} - \lambda| < 2^{-M_0}$ for any $\lambda \in \mathbb{Q} \cap (0, 1)$.*

Remark 1. Even though the statement $|\Omega_{A_*} - \lambda| < 2^{-M_0}$ is true for a countably infinite subset of $\mathbb{Q} \cap (0, 1)$, it cannot be proved for a single of these rational numbers.

Proof. Suppose that the statement of the theorem is false. Then, for every $M \in \mathbb{N}$, ZFC can determine a $\lambda_M \in \mathbb{Q}$ such that $|\Omega_{A_*} - \lambda_M| < 2^{-M}$. λ_M is in $\mathbb{Q}$, i.e., we have $\lambda_M = p_M/q_M$ for some $p_M, q_M \in \mathbb{N}$, where we assume that p_M and q_M are coprime. Hence, we can determine $d_M = |1/2 - \lambda_M| = |q_M - 2p_M|/(2q_M)$. $\{d_M\}_{M\in\mathbb{N}}$ is a sequence of rational numbers. Let M_1 be the smallest natural number such that $2^{-M_1} < d_{M_1}$. For M_1 we compute $v_{M_1} = q_{M_1} - 2p_{M_1}$. If $v_{M_1} > 0$ then $\lambda_{M_1} < 1/2$, and if $v_{M_1} < 0$ then $\lambda_{M_1} > 1/2$. $v_{M_1} = 0$ cannot occur. Further, if $\lambda_{M_1} < 1/2$ then we have $\Omega_{A_*} < 1/2$, and if $\lambda_{M_1} > 1/2$ then we have $\Omega_{A_*} > 1/2$. Hence we can determine whether $\Omega_{A_*} < 1/2$ or $\Omega_{A_*} > 1/2$. This means we can determine the first binary digit of the binary expansion of Ω_{A_*}, which is a contradiction. □

The observation that for any number that is Turing computable, ZFC can determine every binary digit of the binary expansion, leads to the following corollary.

Corollary 2. *Let f_* be the same function as in Theorem 2. If ZFC is arithmetically sound, then $\hat{f}_*$ is not Turing computable as continuous function, because $\hat{f}_*(0)$ is not Turing computable.*

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