Banach frames and atomic decompositions in the space of bounded operators on Hilbert spaces

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***Abstract*—The concept of frames is used extensively for the representation of signal or functions. Recently, this concept is applied more and more for the representation of operators, both in theory as well as in the application for the numerical solutions of operator equations. In this paper we first give a survey about the matrix representation of operators using frames. Then we prove that the tensor product of frames forms a Banach frame and an atomic decomposition for the space of bounded operators of Hilbert spaces.**

I. Introduction

A very important question in mathematics is how a given object can be represented as sum of certain building blocks. A standard approach is using orthonormal bases, but redundant systems, i.e. frames, have gained a lot of prominence in the last decades. The mathematical concept of *frames* was introduced by [20] and popularized by [19]. These systems can be used to represent functions by discrete samples in a stable and invertible way. Frame theory is now a very active field of mathematical research [1], [13], [30] and has found applications in signal processing [8], quantum mechanics [24], acoustics [6] and various other fields. The standard setting for frames are Hilbert spaces, but, amongst others, it can also be generalized to the Banach space setting [26], [14], [11].

It is very natural to extend this approach to operators. On an abstract level, it is well known that for orthonormal bases operators can be uniquely described by a matrix representation [25]. An analogous result holds for frames and their duals [4], [5]. For a bounded, linear operator O, define the infinite matrix $\langle O\phi_l, \psi_k\rangle$, for $\Psi = (\psi_k)_{k\in K}$ and $\Phi = (\psi_k)_{k\in K}$ being frames in the Hilbert space $\mathcal{H}$. On the theoretical level, there are several results related to the representation of operators by matrices using frames, see e.g. [22], [28], [34]. On the applied side, in the numerical treatment of operator equations this is related to the Galerkin scheme [31]. In the Finite Element Method [9] and the Boundary Element Method [23] usually spline-like bases are used. More recently, wavelet bases [18] and frames [32], [27], [35] have been applied.

In [3], it was shown that for any two frames Ψ and Φ the tensor product $\Psi \otimes \Phi$ forms a frame in the class of Hilbert Schmidt operators $\mathcal{HS}$.

Here we use the same results to show that the tensors form a Banach frame as well as an atomic decomposition in the space of all bounded operators for the Hilbert space $\mathcal{H}$. While being purely a frame theory result, this could be used in the future for the numerical treatment of operator equations in settings, where e.g. some smoothness criterion are based on Banach norms [17] or where a scale of (Banach) spaces are relevant [16].

In Section II we give the preliminaries, in particular details on frame theory. In Section III we give a review of results for the matrix representation of operators using frames. Finally, in Section IV we show that the tensors of two frame systems are a Banach frame and an atomic decomposition for the space of bounded operators.

II. Preliminaries

Denote the canonical basis in ℓ^2 by δ_i.

Let $f \in \mathcal{H}_1$, $g \in \mathcal{H}_2$, define the *tensor product* as the operator from $\mathcal{H}_2$ to $\mathcal{H}_1$ given by $(f \otimes \overline{g})(h) := \langle h, g\rangle f$ for $h \in \mathcal{H}_2$. It is a bounded operator with bound $\|f\|_{\mathcal{H}_1} \cdot \|g\|_{\mathcal{H}_2}$.

A. Frames in Hilbert Spaces

A sequence $\Psi = (\psi_k | k \in K)$ is called a *frame* [20], [12], [10] for the Hilbert space $\mathcal{H}$, if there exist constants $A_\Psi, B_\Psi > 0$, such that

$$A_\Psi \cdot \|f\|_{\mathcal{H}}^2 \leq \sum_k |\langle f, \psi_k\rangle|^2 \leq B_\Psi \cdot \|f\|_{\mathcal{H}}^2 \qquad \forall\ f \in \mathcal{H} \quad (1)$$

Here A_Ψ is called a *lower* and B_Ψ an *upper frame bound*. If we only consider the right inequality such a sequence is called a *Bessel sequence*.

For a Bessel sequence, $\Psi = (\psi_k)$, define the *analysis operator* $C_\Psi : \mathcal{H} \to \ell^2(K)$ by $C_\Psi(f) = (\langle f, \psi_k\rangle)_k$. Let $D_\Psi : \ell^2(K) \to \mathcal{H}$ be the *synthesis operator* $D_\Psi((c_k)) = \sum\limits_k c_k \cdot \psi_k$. Let $S_\Psi : \mathcal{H} \to \mathcal{H}$ be the *(associated) frame operator* $S_\Psi(f) = \sum\limits_k \langle f, \psi_k\rangle \cdot \psi_k$. C_Ψ and D_Ψ are adjoint to each other, $D_\Psi = C_\Psi^*$ with $\|D_\Psi\|_{\ell^2\to\mathcal{H}} = \|C_\Psi\|_{\mathcal{H}\to\ell^2} \leq \sqrt{B}$. The series $\sum\limits_k c_k \cdot \psi_k$ converges unconditionally for all $(c_k) \in \ell^2$.

For a frame $\Psi = (\psi_k)$ the operator C_Ψ is a bounded, injective operator with closed range and $S_\Psi = C_\Psi^* C_\Psi = D_\Psi D_\Psi^*$ is a positive invertible operator satisfying $A_\Psi I_{\mathcal{H}} \leq S_\Psi \leq B_\Psi I_{\mathcal{H}}$ and $B_\Psi^{-1} I_{\mathcal{H}} \leq S_\Psi^{-1} \leq A_\Psi^{-1} I_{\mathcal{H}}$. Even more, we can find an expansion for every member of $\mathcal{H}$: The sequence $\tilde{\Psi} = \left(\tilde{\psi}_k\right) = \left(S_\Psi^{-1}\psi_k\right)$ is a frame with frame bounds B_Ψ^{-1}, $A_\Psi^{-1} > 0$, the so called *canonical dual frame*. Every $f \in \mathcal{H}$ has the expansions

$f = \sum_{k \in K} \left\langle f, \tilde{\psi}_k \right\rangle \psi_k$ and $f = \sum_{k \in K} \langle f, \psi_k \rangle \tilde{\psi}_k$ where both sums converge unconditionally in $\mathcal{H}$.

Two sequences (ψ_k), (ϕ_k) are called *biorthogonal* if $\langle \psi_k, \phi_j \rangle = \delta_{kj}$ for all h, j.

A sequence (ψ_k) in $\mathcal{H}$ is called a *Riesz sequence* if there exist constants A_Ψ, $B_\Psi > 0$ such that the inequalities

$$A_\Psi \|c\|_2^2 \leq \left\| \sum_{k \in K} c_k \psi_k \right\|_{\mathcal{H}}^2 \leq B_\Psi \|c\|_2^2$$

hold for all finite sequences (c_k). It is called a *Riesz basis*, if it is complete as well.

For a frame (ψ_k) the following conditions are equivalent:

(i) (ψ_k) is a Riesz basis for $\mathcal{H}$.
(ii) The coefficients $(c_k) \in \ell^2$ for the series expansion with (ψ_k) are unique, i.e. the synthesis operator D_Ψ is injective.
(iii) The analysis operator C_Ψ is surjective.
(iv) (ψ_k) and $(\tilde{\psi}_k)$ are biorthogonal.

The *Gram matrix* $G_{\Psi,\Phi}$ is given by $(G_{\Psi,\Phi})_{j,m} = \langle \phi_m, \psi_j \rangle$, $j, m \in K$. Therefore as an operator from ℓ^2 into ℓ^2 it is given by $G_{\Psi,\Phi} = C_\Psi \circ D_\Phi$. For a frame $G_{\Psi,\tilde{\Psi}}$ represents the projection on $\mathsf{ran}\,(C_\Psi)$, denoted by $\Pi_{\mathsf{ran}(C_\Psi)}$.

B. Banach frames

The concept of frames can be extended to Banach spaces [26], [14], [11]:

Let X be a Banach space and X_d be a Banach space of scalar sequences. A sequence (ψ_k) in the dual X' is called a X_d-frame for the Banach space X if there exist constants $A_\Psi, B_\Psi > 0$ such that

$$A_\Psi \|f\|_X \leq \|\langle f, \psi_k \rangle_{k \in K}\|_{X_d} \leq B_\Psi \|f\|_X \quad \text{for all} \quad f \in X. \tag{2}$$

An X_d-frame is called a *Banach frame* with respect to a sequence space X_d, if there exists a bounded reconstruction operator $R : X_d \to X$, such that $R(\psi_k(f)) = f$ for all $f \in X$.

On the other hand the pair (ψ_k, ϕ_k) is called an *atomic decomposition* of X with respect to X_d if Ψ is a Banach frame and $\Phi := (\phi_k) \subseteq X$ and $f = \sum_{k=1}^\infty \psi_k(f) \phi_k, \forall f \in X$.

III. Matrix Representation using Frames

It is well known that operators can be uniquely described by a matrix representation [25] using fixed orthonormal bases. The same can be constructed with frames and their duals, see [4]. Note that we will use the notation $\|.\|_{\mathcal{H}_1 \to \mathcal{H}_2}$ for the operator norm in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ to be able to distinguish between different operator norms.

A. Operators on Hilbert Spaces

Theorem III.1. *Let $\Psi = (\psi_k)$ be a frame in $\mathcal{H}_1$, $\Phi = (\phi_k)$ in $\mathcal{H}_2$.*

1) *Let $O : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded, linear operator. Then the infinite matrix*

$$\left(\mathcal{M}^{(\Phi,\Psi)}(O) \right)_{m,n} = \langle O \psi_n, \phi_m \rangle_{\mathcal{H}_2}$$

defines a bounded operator from ℓ^2 to ℓ^2 with

$$\left\| \mathcal{M}^{(\Phi,\Psi)}(O) \right\|_{\ell^2 \to \ell^2} \leq \sqrt{B_\Phi \cdot B_\Psi} \cdot \|O\|_{\mathcal{H}_1 \to \mathcal{H}_2}. \tag{3}$$

As an operator $\ell^2 \to \ell^2$

$$\mathcal{M}^{(\Phi,\Psi)}(O) = C_\Phi \circ O \circ D_\Psi$$

This means that the function

$$\mathcal{M}^{(\Phi,\Psi)} : \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \to \mathcal{B}\left(\ell^2, \ell^2\right)$$

is a well-defined bounded operator.

2) *On the other hand let M be an infinite matrix defining a bounded operator from ℓ^2 to ℓ^2, $(Mc)_i = \sum_k M_{i,k} c_k$. Then the operator $\mathcal{O}^{(\Phi,\Psi)}$ defined by*

$$\left(\mathcal{O}^{(\Phi,\Psi)}(M) \right) h = \sum_k \left(\sum_j M_{k,j} \langle h, \psi_j \rangle \right) \phi_k,$$

for $h \in \mathcal{H}_1$. This is a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ with

$$\left\| \mathcal{O}^{(\Phi,\Psi)}(M) \right\|_{\mathcal{H}_1 \to \mathcal{H}_2} \leq \sqrt{B_\Psi \cdot B_\Psi} \|M\|_{\ell^2 \to \ell^2}. \tag{4}$$

We have

$$\mathcal{O}^{(\Phi,\Psi)}(M) = D_\Phi \circ M \circ C_\Psi = \sum_k \sum_j M_{k,j} \cdot \phi_k \otimes_i \overline{\psi}_j$$

This means the function $\mathcal{O}^{(\Phi,\Psi)} : \mathcal{B}\left(\ell^2, \ell^2\right) \to \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is a well-defined bounded operator.

For frames more properties can be proved [4]:

Proposition III.2. *Let $\Psi = (\psi_k)$ be a frame in $\mathcal{H}_1$, $\Phi = (\phi_k)$ in $\mathcal{H}_2$. Then we have*

$$\mathsf{id}_{\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)} = \left(\mathcal{O}^{(\tilde{\Phi},\tilde{\Psi})} \circ \mathcal{M}^{(\Phi,\Psi)} \right). \tag{5}$$

And therefore for all $O \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$:

$$O = \sum_{k,j} \langle O \psi_j, \phi_k \rangle \tilde{\phi}_k \otimes \overline{\tilde{\psi}}_j. \tag{6}$$

We have that $\mathcal{M}^{(\Phi,\Psi)}$ is injective and $\mathcal{O}^{(\Phi,\Psi)}$ is surjective. Let $\mathcal{H}_1 = \mathcal{H}_2$, then $\mathcal{O}^{(\Psi,\tilde{\Psi})}(Id_{\ell^2}) = \mathsf{id}_{\mathcal{H}_1}$ Let $\Xi = (\xi_k)$ be any frame in $\mathcal{H}_3$, and $O : \mathcal{H}_3 \to \mathcal{H}_2$ and $P : \mathcal{H}_1 \to \mathcal{H}_3$. Then

$$\mathcal{M}^{(\Phi,\Psi)}(O \circ P) = \left(\mathcal{M}^{(\Phi,\Xi)}(O) \cdot \mathcal{M}^{(\tilde{\Xi},\Psi)}(P) \right)$$

The other operator $\mathcal{O}$ is, in general, not multiplicative in the above sense. It can even been shown that if it is for all matrices, then the involved sequences must be Riesz bases [7]. But we can state the following result:

Theorem III.3. *[7] Let Φ, Ξ, and Ψ be frames for $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ respectively, and $M^{(1)}$ and $M^{(2)}$ be in $\mathcal{B}\left(\ell^2\right)$. Then if*

(a) $M^{(2)}(\mathsf{ran}\,(C_\Psi)) \subseteq \mathsf{ran}\,(C_\Xi)$ *, or*
(b) ${M^{(1)}}^*(\mathsf{ran}\,(C_\Phi)) \subseteq \mathsf{ran}\,(C_\Xi)$ *,*

we get that

$$\mathcal{O}^{(\Phi,\Psi)}\left(M^{(1)} \cdot M^{(2)} \right) =$$
$$\mathcal{O}^{(\Phi,\Xi)}\left(M^{(1)} \right) \circ \mathcal{O}^{(\tilde{\Xi},\Psi)}\left(M^{(2)} \right).$$

B. Tensor of Frames in $\mathcal{HS}$

In [3] the following was proved:

Theorem III.4. *Let (ϕ_k), (ψ_i) be sequences in $\mathcal{H}_2$ resp. $\mathcal{H}_1$.*

1) *If (ϕ_k) and (ψ_i) are Bessel sequences, then $(\psi_i \otimes \overline{\phi}_k)$ is a Bessel sequence for $\mathcal{HS}(\mathcal{H}_2, \mathcal{H}_1)$ with bound $B_\Phi \cdot B_\Psi$.*
2) *If (ϕ_k) and (ψ_i) are frames, then $(\psi_i \otimes \overline{\phi}_k)$ is a frame for $\mathcal{HS}(\mathcal{H}_2, \mathcal{H}_1)$ with bounds $A_\Phi \cdot A_\Psi$ and $B_\Phi \cdot B_\Psi$. The analysis operator is*
$$C_{\psi_i \otimes \overline{\phi}_k}(O) = \mathcal{M}^{(\psi_k, \phi_j)}(O)_{k,j} = \left(\langle O\phi_j, \psi_k \rangle_{\mathcal{H}_1} \right).$$
The synthesis operator is
$$D_{\psi_i \otimes \overline{\phi}_k}(M) = \mathcal{O}^{(\psi_k, \phi_j)}(M) = \sum_{k,j} M_{k,j}\ \phi_k \otimes \psi_j.$$
The frame operator is $S_{\psi_i \otimes \overline{\phi}_k} = S_{\psi_i} \otimes S_{\phi_k}$, meaning that $S_{\psi_i \otimes \overline{\phi}_k}(O) = S_{\psi_i} \circ O \circ S_{\phi_k}$. The canonical dual frame is
$$\left(\widetilde{\psi_i \otimes \overline{\phi}_k}\right) = \left(\widetilde{\psi}_i \otimes \overline{\widetilde{\phi}}_k\right).$$
3) *If (ϕ_k) and (ψ_i) are Riesz bases, then $(\psi_i \otimes \overline{\phi}_k)$ is a Riesz basis for $\mathcal{HS}(\mathcal{H}_2, \mathcal{H}_1)$. The biorthogonal sequence is $(\tilde{\psi}_i \otimes \overline{\tilde{\phi}}_k)$.*

This can easily be extended to more operator spaces. In the next section we prove results for the space of bounded operators, in the upcoming [7] we will deal with Schatten-p-class operators, resulting in Gelfand triplets and scales of Banach spaces.

IV. Tensors of Frames in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$

The space of bounded operators on ℓ^2 can be classified, see [15], [29]. For this section it is important that $\mathcal{B}(\ell^2, \ell^2)$ is a Banach space of matrices and therefore a sequence space. For some properties regarding Banach frames [11], [33] particular classes of sequence are considered: a Banach sequence space is called a *BK-space* if the coordinate functionals are continuous. It is called a *CB-space*, if the canonical vectors form a Schauder basis. It is called a *RCB-space* if it is a reflexive CB-space.

In $\mathcal{B}(\ell^2, \ell^2)$ the canconical vectors are a Schauder basis, where we consider the canonical matrices $E_{i,j} = \delta_i \cdot \delta_j$ [2]. The coordinate functionals are continuous. Indeed, assume
$$\|M^n - M\|_{\mathcal{B}(\ell^2, \ell^2)} \to 0.$$
We have that
$$|M^n_{i,j} - M_{i,j}| = |\langle (M^n - M)\delta_i, \delta_j \rangle| \leq$$
$$\leq \|(M^n - M)\delta_i\|_2 \|\delta_j\|_2 \leq \|M^n - M\|_{\mathcal{B}(\ell^2, \ell^2)}.$$

It follows that $\mathcal{B}(\ell^2, \ell^2)$ is a BK-space and a CB-space, but not an RCB-space (as we know that the predual is not the dual space.)

The operator $f \otimes g$ is not only an element in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ but can also be considered as element of the dual of this space. We set, see e.g. [21], $\langle O, f \otimes \overline{g} \rangle_{\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2), \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)'} := \langle Og, f \rangle_{\mathcal{H}_2}$ for an $O \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. By using the space of Hilbert space operators as motivation we show that this is a very natural connection, as for any ONB (e_k) we have
$$\langle O, f \otimes \overline{g} \rangle_{\mathcal{HS}} = \sum_k \langle Oe_k, (f \otimes \overline{g})\, e_k \rangle =$$
$$= \sum_k \langle Oe_k, f \rangle \langle e_k, g \rangle = \langle Og, f \rangle_{\mathcal{H}_2}.$$
This was already used in [3] e.g. for Theorem III.4.

In summary we can show

Proposition IV.1. *Let Ψ and Φ be frames in $\mathcal{H}_1$ respectively $\mathcal{H}_2$. Then $\Psi \otimes \Phi := \left(\psi_k \otimes \overline{\phi_l}\right)_{(k,l) \in K \times K}$ is a X_d-frame for $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ for the BK and CB-space of sequences $X_d = \mathcal{B}(\ell^2, \ell^2)$. The bounds are $\sqrt{A_\Phi \cdot A_\Psi}$ and $\sqrt{B_\Phi \cdot B_\Psi}$.*

Proof. By the comments above we have clarified the properties of the sequence space X.

By (3) we have the upper bound $\sqrt{B_\Psi \cdot B_\Phi}$.

By (5) we have
$$O = \mathcal{O}^{(\tilde{\Phi}, \tilde{\Psi})}\left(\mathcal{M}^{(\Phi, \Psi)}(O)\right).$$
By (4)
$$\|O\|_{\mathcal{H}_1 \to \mathcal{H}_2} \leq \sqrt{\frac{1}{A_\Psi} \cdot \frac{1}{A_\Phi}} \left\|\mathcal{M}^{(\Phi, \Psi)}(O)\right\|_{\ell^2 \to \ell^2},$$
and therefore in summary
$$\sqrt{A_\Psi \cdot A_\Phi}\, \|O\|_{\mathcal{H}_1 \to \mathcal{H}_2} \leq \left\| \langle O, \psi_k \otimes \phi_k \rangle_{\mathcal{B}(X_1, X_2), \mathcal{B}(X_1, X_2)'} \right\|_{X_d}$$
$$\leq \sqrt{B_\Psi \cdot B_\Phi}\, \|O\|_{\mathcal{H}_1 \to \mathcal{H}_2}.$$
□

We will now investigate the Banach frame properties of the set $\{\psi_k \otimes \phi_k\}_{(k,l) \in K \times K}$.

Corollary IV.2. *Let Ψ and Φ be frames in $\mathcal{H}_1$ respectively $\mathcal{H}_2$. Then $\Psi \otimes \Phi := (\psi_k \otimes \phi_l)_{(k,l) \in K \times K}$ is a Banach frame for $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$.*

Proof. The reconstruction is just (5), so the reconstruction operator is $\mathcal{O}^{((\widetilde{\Phi}, \widetilde{\Psi}))}$. □

If we now consider $\Psi \otimes \Phi$ as element in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ we can show

Corollary IV.3. *Let Ψ and Φ be frames in $\mathcal{H}_1$ respectively $\mathcal{H}_2$. Then the pair $\left(\Psi \otimes \Phi, \widetilde{\Psi} \otimes \widetilde{\Phi}\right)$ is an atomic decomposition for $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$.*

Proof. This is just Prop. IV.1 and a rephrasing of (6):
$$O = \sum_{k,j} \langle O, \psi_k \otimes \phi_l \rangle_{\mathcal{B}(X_1, X_2), \mathcal{B}(X_1, X_2)'}\, \tilde{\psi}_k \otimes \overline{\tilde{\psi}}_l.$$
□

Please note that similar arguments are valid if we consider Banach spaces instead of Hilbert spaces.

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