Fast Multitaper Spectral Estimation

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Abstract—Thomson’s multitaper method using discrete prolate spheroidal sequences (DPSSs) is a widely used technique for spectral estimation. For a signal of length \( N \), Thomson’s method requires selecting a bandwidth parameter \( W \), and then uses \( K \approx 2NW \) tapers. The computational cost of evaluating the multitaper estimate at \( N \) grid frequencies is \( O(KN \log N) \). It has been shown that the choice of \( W \) and \( K \) which minimizes the MSE of the multitaper estimate is \( W = O(N^{-1/2}) \) and \( K = O(N^{4/5}) \). This choice would require a computational cost of \( O(N^{9/5} \log N) \). We demonstrate an \( \epsilon \)-approximation to the multitaper estimate which can be evaluated at \( N \) grid frequencies using \( O(N \log^2 N \log \frac{1}{\epsilon}) \) operations.

I. INTRODUCTION

Let \( x(t), t \in \mathbb{R} \) be a stationary, ergodic, zero-mean, Gaussian stochastic process. The Cramer representation of \( x(t) \) is given by

\[
x(t) = \int_{-1/2}^{1/2} e^{j2\pi ft} dZ(f),
\]

and the spectral density of \( x(t) \) is given by

\[
S(f) df = \mathbb{E} \left[ |dZ(f)|^2 \right].
\]

The problem of spectral estimation is to estimate \( S(f) \) from \( N \) equally spaced samples

\[
x = [x(0) \ x(1) \ \cdots \ x(N-1)]^T \in \mathbb{C}^N.
\]

Thomson’s multitaper method for spectral estimation [7] can be described as follows. For a given half-bandwidth parameter \( W \in (0, \frac{1}{2}) \), we define the Slepian basis vectors \( s_0, s_1, \ldots, s_{N-1} \in \mathbb{R}^N \) as the orthonormal eigenvectors of the \( N \times N \) prolate matrix \( \mathbf{B} \), whose entries are given by

\[
\mathbf{B}[m,n] = \frac{\sin[2\pi W(m-n)]}{\pi(m-n)} \quad \text{for } m,n \in [N].
\]

The eigenvectors are ordered such that corresponding eigenvalues \( \lambda_0 > \lambda_1 > \cdots > \lambda_{N-1} \) are sorted in descending order. For each \( k \in [N] \), we can use \( s_k \) as a taper to define a single tapered spectral estimate \( \hat{S}_k(f) \), i.e.,

\[
\hat{S}_k(f) = \left| \sum_{n=0}^{N-1} s_k[n] x[n] e^{-j2\pi fn} \right|^2.
\]

Then, we pick an integer \( K \) and define the unweighted multitaper spectral estimate of \( x \) as

\[
\hat{S}_K^u(f) = \frac{K}{K-1} \sum_{k=0}^{K-1} \hat{S}_k(f).
\]

Since the first slightly less than 2NW Slepian basis vectors have spectra concentrated in \([-W, W]\), the number of tapers \( K \) is usually chosen to be slightly less than 2NW. Thomson also considered the eigenvalue weighted multitaper spectral estimate [7]

\[
\hat{S}_K^w(f) = \sum_{k=0}^{K-1} \frac{\lambda_k \bar{S}_k(f)}{\sum_{k=0}^{K-1} \lambda_k}.
\]

In many applications, it is desirable to estimate the spectrum on a grid of \( N \) evenly spaced frequencies, i.e., \( f = \frac{m}{N} \) for \( m \in [N] \). For each \( k \in [K] \), evaluating \( \hat{S}_k(f) \) at all \( N \) grid frequencies takes \( O(KN \log N) \) operations via a length-\( N \) FFT of the elementwise product \( s_k \circ \varphi \). After this, only \( O(KN) \) more operations are needed to evaluate the weighted/unweighted sum at all \( N \) grid frequencies. Hence, the total computation required to evaluate either \( \hat{S}_K^w(f) \) or \( \hat{S}_K^w(f) \) at the \( N \) grid frequencies can be done in \( O(KN \log N) \) operations. Also, the cost of precomputing the tapers \( s_0, \ldots, s_{K-1} \) is \( O(KN \log N) \) operations. This is due to the fact that there is a tridiagonal matrix \( \mathbf{T} \) with the same eigenvectors \( s_0, s_1, \ldots, s_{N-1} \) as \( \mathbf{B} \) and similarly ordered eigenvalues \( \mu_0 > \mu_1 > \cdots > \mu_{N-1} \) [6].

In [8], it is shown that if \( S(f) \) is twice differentiable, then bias and variance of \( \hat{S}_K^u(f) \) are bounded by

\[
\text{Bias} \left( \hat{S}_K^w(f) \right) \lesssim \frac{W^2}{6} S''(f),
\]

\[
\text{Var} \left( \hat{S}_K^w(f) \right) \lesssim \frac{1}{K} S(f)^2;
\]

and thus, the mean-squared error is bounded by

\[
\text{MSE} \left( \hat{S}_K^w(f) \right) \lesssim \frac{W^4}{36} S''(f)^2 + \frac{1}{K} S(f)^2.
\]

Since \( K \approx 2NW \), this bound is minimized when

\[
W \sim \left[ \frac{9S(f)}{25S''(f)} \right]^{2/5} N^{-1/5} \quad \text{and} \quad K \sim \left[ \frac{12S(f)}{S''(f)} \right]^{2/5} N^{4/5}.
\]

Similar analysis is done in [5] for sinusoidal tapers and in [1] for Slepian tapers. In general, fewer tapers are used for more rapidly varying spectra, but for any fixed spectrum \( S(f) \) and for large \( N \), the optimal number of tapers is \( K = O(N^{4/5}) \). However, this choice requires precomputing \( O(N^{4/5}) \) tapers and then \( O(N^{4/5}) \) length-\( N \) FFTs to evaluate \( \hat{S}_K(f) \) at all \( N \) grid frequencies. This involves \( O(N^{9/5} \log N) \) operations.
In this work, we present approximations \( \hat{S}_{mK}^f \) and \( \hat{S}_{K}^f \) to \( S_{mK}^f \) and \( S_K^f \) respectively which satisfy
\[
\left| \hat{S}_{mK}^f (f) - \hat{S}_{K}^f (f) \right| \leq \frac{O(\epsilon)}{K} \| x \|_2^2 \quad \text{for all } f \in \mathbb{R},
\]
and which can be evaluated at all grid frequencies in \( O(N \log^2 N \log \frac{1}{\epsilon}) \) operations. Also, the required precomputation for these approximations takes only \( O(N \log^2 N \log \frac{1}{\epsilon}) \) operations. When the number of tapers is \( K \gtrsim \log N \log \frac{1}{\epsilon} \), evaluating \( \hat{S}_{mK}^f(f) \) at the \( N \) grid frequencies will be significantly faster than evaluating \( \hat{S}^f_{mK} \) at the \( N \) grid frequencies.

### II. Intermediate Results

#### A. Fast algorithm for computing \( \hat{S}_{mK}^f \)

To begin developing our fast approximations for \( \hat{S}^f_{mK} \) and \( \hat{S}^f_{K} \), we first consider the eigenvalue weighted multitaper spectral estimate with \( N \) tapers instead of \( K \approx 2NW \), i.e.,
\[
\hat{S}^f_{mK}(f) = \frac{1}{2NW} \sum_{k=0}^{N-1} \lambda_k \hat{S}_k(f).
\]
Using an eigendecomposition, we can write \( B = SAS^* \), where \( S = [s_0 \ldots s_{N-1}] \) and \( A = \text{diag}(\lambda_0, \ldots, \lambda_{N-1}) \).

For any \( f \in \mathbb{R} \), let \( E_f \in \mathbb{C}^{N \times N} \) be a diagonal matrix with diagonal entries \( E_f[n,n] = e^{j2\pi fn} \). Then, \( \hat{S}^f_{mK} \) satisfies
\[
2NW \hat{S}^f_{mK}(f) = \sum_{k=0}^{N-1} \lambda_k \hat{S}_k(f) = \sum_{k=0}^{N-1} \lambda_k \left| \sum_{n=0}^{N-1} s_k[n] x[n] e^{-j2\pi fn} \right|^2 = \sum_{k=0}^{N-1} \lambda_k |s_k E_f x|^2 = x^* E_f S A S^* E_f^* x = x^* E_f B E_f^* x.
\]

This gives us a a formula for \( \hat{S}^f_{mK} \) which does not require computing any of the Slepian tapers. Furthermore, we state a formula that allows us to evaluate \( \hat{S}^f_{mK} \) for all \( m \in \mathbb{N} \) in \( O(N \log N) \) operations.

First we define a vector of sinc samples
\[
b[l] = \begin{cases} 
sin[2\pi W l] & \ell \in [N], \\
0 & \ell = N, \\
\sin[2\pi W (2N - \ell)] & \ell \in [2N] \setminus [N+1], \end{cases}
\]
a zeropadding matrix
\[
Z = \begin{bmatrix} I_{N \times N} \\ 0_{N \times N} \end{bmatrix},
\]
a length-2N FFT matrix defined by
\[
F[m,n] = e^{-j\pi mn/N} \text{ for } m,n \in [2N],
\]
and a vector
\[
y = F^{-1} \left( b \circ F \right) \| F Z x \|^2_2,
\]
where we use the notation \( \circ \) to be the elementwise product, i.e., \( (p \circ q)[\ell] = p[\ell] q[\ell] \), and \( |x|^2 \) to denote the elementwise magnitude-squared, i.e., \( (|p|^2)[\ell] = |p[\ell]|^2 \).

With these definitions, we have \( \hat{S}^f_{mK} = \frac{1}{2NW} y[2m] \) for all \( m \in [N] \). The derivation of this formula involves exploiting the fact that \( B \) is a Toeplitz matrix, and thus, can be extended to a circulant matrix which is diagonalized by \( F \). Due to space limitations, the proof of this formula is deferred to a future publication. Computing \( y = F^{-1} \left( b \circ F \right) \| F Z x \|^2_2 \) can be done in \( O(N \log N) \) operations via three length-2N FFTs and a few pointwise multiplications of length-2N vectors. Then, we can obtain \( \hat{S}^f_{mK} = \frac{1}{2NW} y[2m] \) for \( m \in [N] \) by downsampling and scaling \( z \).

#### B. Approximations for General Multitaper Spectral Estimates

Next, we present a lemma regarding approximations to spectral estimates which use orthonormal tapers.

**Lemma 1.** Let \( x \in \mathbb{C}^N \) be a vector of \( N \) equispaced samples, and let \( \{v_k\}_{k=0}^{N-1} \) be any orthonormal set of tapers in \( \mathbb{C}^N \). For each \( k \in [N] \), define a tapered spectral estimate
\[
V_k(f) = \sum_{n=0}^{N-1} v_k[n] x[n] e^{-j2\pi fn}. 
\]
Also, let \( \{\gamma_k\}_{k=0}^{N-1} \) and \( \{\bar{\gamma}_k\}_{k=0}^{N-1} \) be real coefficients, and then define a multitaper spectral estimate \( \hat{V}(f) \) and an approximation \( \tilde{V}(f) \) by
\[
\hat{V}(f) = \sum_{k=0}^{N-1} \gamma_k V_k(f) \quad \text{and} \quad \tilde{V}(f) = \sum_{k=0}^{N-1} \bar{\gamma}_k V_k(f).
\]

Then, for any frequency \( f \in \mathbb{R} \), we have
\[
\left| \tilde{V}(f) - \hat{V}(f) \right| \leq \left( \max_k |\gamma_k - \bar{\gamma}_k| \right) \|x\|_2^2.
\]

**Proof.** Let \( V = [v_0 \ldots v_{N-1}] \), and let \( \Gamma, \bar{\Gamma} \in \mathbb{R}^{N \times N} \), and \( E_f \in \mathbb{C}^{N \times N} \) be diagonal matrices whose diagonal entries have modulus \( 1 \), \( \| E_f \| = \| E_f^* \| = 1 \). Hence, for any \( f \in \mathbb{R} \), we can bound
\[
\left| \tilde{V}(f) - \hat{V}(f) \right| = \left| x^* E_f V (\Gamma - \bar{\Gamma}) V^* E_f^* x \right| \leq \|x\|_2 \|E_f\| \|V\| \| \Gamma - \bar{\Gamma} \| \|V^*\| \|E_f\| \|x\|_2 = \left( \max_k |\gamma_k - \bar{\gamma}_k| \right) \|x\|_2^2,
\]
as desired.
C. Prolate matrix eigenvalue behavior

The eigenvalues \( \lambda_0 > \lambda_1 > \cdots > \lambda_{N-1} \) of \( B \) are all strictly between 0 and 1, and they have a clustering behavior. For fixed \( W \in (0, \frac{1}{2}) \) and \( \epsilon \in (0, \frac{1}{2}) \) and large \( N \), slightly less than \( 2NW \) eigenvalues are between \( 1 - \epsilon \) and 1, slightly less than \( N - 2NW \) eigenvalues are between 0 and \( \epsilon \), and very few eigenvalues are between \( \epsilon \) and \( 1 - \epsilon \). In [3], it is shown that for fixed \( W \in (0, \frac{1}{2}) \) and \( \epsilon \in (0, \frac{1}{2}) \),

\[
\# \{ k : \epsilon < \lambda_k < 1 - \epsilon \} \sim \frac{2}{\pi^2} \log N \log \left( \frac{1}{\epsilon} - 1 \right)
\]
as \( N \to \infty \). Also, for any \( N \in \mathbb{N} \), \( W \in (0, \frac{1}{2}) \) and \( \epsilon \in (0, \frac{1}{2}) \),

\[
\# \{ k : \epsilon < \lambda_k < 1 - \epsilon \} \leq \left( \frac{8}{\pi^2} \log (8N) + 12 \right) \log \left( \frac{15}{\epsilon} \right).
\]

We assume that for a given \( N \in \mathbb{N} \) and \( W \in (0, \frac{1}{2}) \), the parameters \( K \) and \( \epsilon \in (0, \frac{1}{2}) \) are chosen such that \( \lambda_{K-1} \geq \frac{1}{2} \) and \( \lambda_k \leq 1 - \epsilon \). This restriction only forces \( K \) to be slightly less than \( 2NW \). We partition the indices \([N]\) into four sets

\[
\begin{align*}
I_1 &= \{ k \in [K] : \lambda_k \geq 1 - \epsilon \}, \\
I_2 &= \{ k \in [K] : \epsilon < \lambda_k < 1 - \epsilon \}, \\
I_3 &= \{ k \in [N] \setminus [K] : \epsilon < \lambda_k < 1 - \epsilon \}, \\
I_4 &= \{ k \in [N] \setminus [K] : \lambda_k \leq \epsilon \}.
\end{align*}
\]

In the following subsections, we will assume that we have precomputed \( \lambda_k \) and \( s_k \) for all \( k \in I_2 \cup I_3 \), but not necessarily for any \( k \in I_1 \cup I_4 \). This can be done in \( O(N \log^2 N \log \frac{1}{\epsilon}) \) operations as follows. First, bound the “transition region” by \( I_2 \cup I_3 \subseteq \llbracket \lceil 2NW \rceil + r \rrbracket \setminus \llbracket \lceil 2NW \rceil - r \rrbracket =: I \) where

\[
\#(I_2 \cup I_3) \leq r = O(\log N \log \frac{1}{\epsilon}).
\]

Then, use a bisecion and inverse iteration algorithm [2] on the tridiagonal matrix \( T \) from [6] which has the same eigenvectors \( s_0, s_1, \ldots, s_{N-1} \) as \( B \) and similarly ordered eigenvalues \( \mu_0 > \mu_1 > \cdots > \mu_{N-1} \). Only \( \mu_k \) and \( s_k \) for \( k \in I \) need to be computed, so this takes \( O(2mN) = O(N \log N \log \frac{1}{\epsilon}) \) operations. Finally, for each \( k \in I \), compute \( \lambda_k = s_k^* B s_k \). This takes \( O(2m \log N) = O(N \log^2 N \log \frac{1}{\epsilon}) \) operations since \( B \) is Toeplitz.

III. FAST APPROXIMATIONS

A. Fast algorithm for approximating \( \tilde{S}_{K}(f) \)

The unweighted multitaper spectral estimate \( \tilde{S}_{K}(f) \) is given by

\[
\tilde{S}_{K}(f) = \frac{1}{K} \sum_{k=0}^{K-1} \tilde{S}_k(f) = \frac{1}{K} \sum_{k \in I_1 \cup I_2} \tilde{S}_k(f).
\]

We then define an approximation by

\[
\tilde{S}_{K}(f) := \frac{2NW}{K} \tilde{S}_{N}(f) + \sum_{k \in I_2} \frac{1 - \lambda_k}{K} \tilde{S}_k(f) - \sum_{k \in I_3} \frac{\lambda_k}{K} \tilde{S}_k(f)
\]

\[
= \sum_{k=0}^{N-1} \frac{\lambda_k}{K} \tilde{S}_k(f) + \sum_{k \in I_2} \frac{1 - \lambda_k}{K} \tilde{S}_k(f) - \sum_{k \in I_3} \frac{\lambda_k}{K} \tilde{S}_k(f)
\]

\[
= \sum_{k \in I_1 \cup I_2} \frac{\lambda_k}{K} \tilde{S}_k(f) + \sum_{k \in I_2} \frac{1}{K} \tilde{S}_k(f).
\]

Thus, \( \tilde{S}_{K}^m(f) \) and \( \tilde{S}_{K}^m(f) \) can be written as

\[
\tilde{S}_{K}^m(f) = \sum_{k=0}^{N-1} \gamma_k^m \tilde{S}_k(f) \quad \text{and} \quad \tilde{S}_{K}^m(f) = \sum_{k=0}^{N-1} \gamma_k^m \tilde{S}_k(f)
\]

where

\[
\gamma_k^m = \begin{cases} \frac{1}{K} & k \in I_1 \cup I_2, \\
0 & k \in I_3 \cup I_4,
\end{cases} \quad \text{and} \quad \gamma_k^m = \begin{cases} \frac{\lambda_k}{K} & k \in I_1 \cup I_2, \\
0 & k \in I_3.
\end{cases}
\]

We now consider \( \text{gap}_k^m := |\gamma_k^m - \gamma_k^m| \). For \( k \in I_1 \), we have \( \lambda_k \geq 1 - \epsilon \), and thus,

\[
\text{gap}_k^m = \left| \frac{1}{K} - \frac{\lambda_k}{K} \right| = \left| 1 - \frac{\lambda_k}{K} \right| \leq \epsilon.
\]

For \( k \in I_2 \cup I_3 \) we have \( \gamma_k^m = \gamma_k^m \), i.e., \( \text{gap}_k^m = 0 \). For \( k \in I_4 \), we have \( \lambda_k \leq \epsilon \), and thus,

\[
\text{gap}_k^m = \left| 0 - \frac{\lambda_k}{K} \right| = \frac{\lambda_k}{K} \leq \epsilon.
\]

Hence, \( \text{gap}_k^m \leq \frac{\epsilon}{K} \) for all \( k \in [N] \), and thus by Lemma 1,

\[
\left| \tilde{S}_{K}^m(f) - \tilde{S}_{K}^m(f) \right| \leq \epsilon \| \mathbf{x} \|^2_2.
\]

Finally, evaluating the approximation

\[
\tilde{S}_{K}^m(f) := \frac{2NW}{K} \tilde{S}_{N}(f) + \sum_{k \in I_2} \frac{1 - \lambda_k}{K} \tilde{S}_k(f) - \sum_{k \in I_3} \frac{\lambda_k}{K} \tilde{S}_k(f)
\]
at the \( N \) grid frequencies requires evaluating \( \tilde{S}_{N}(f) \) and \( \tilde{S}_k(f) \) for all \( k \in I_2 \cup I_3 \) at the \( N \) grid frequencies. Evaluating \( \tilde{S}_{N}(f) \) at the grid frequencies takes \( O(N \log N) \) operations, as shown in Section II-A. For each \( k \in I_2 \cup I_3 \), evaluating \( \tilde{S}_k(f) \) at the grid frequencies takes \( O(N \log N) \) operations. Since \( \#(I_2 \cup I_3) = O(\log N \log \frac{1}{\epsilon}) \), the total computation required is \( O(N \log^2 N \log \frac{1}{\epsilon}) \) operations.

B. Fast algorithm for approximating \( \tilde{S}_{K}^e(f) \)

The eigenvalue weighted multitaper spectral estimate \( \tilde{S}_{K}^e(f) \) is given by:

\[
\tilde{S}_{K}^e(f) = \frac{1}{\Sigma_K} \sum_{k=0}^{K-1} \lambda_k \tilde{S}_k(f) = \frac{1}{\Sigma_K} \sum_{k \in I_1 \cup I_2} \lambda_k \tilde{S}_k(f).
\]

We then define an approximation by

\[
\tilde{S}_{K}^e(f) := \frac{2NW}{\Sigma_K} \tilde{S}_{N}(f) - \frac{1}{\Sigma_K} \sum_{k \in I_3} \lambda_k \tilde{S}_k(f)
\]

\[
= \frac{1}{\Sigma_K} \sum_{k \in I_1 \cup I_2} \lambda_k \tilde{S}_k(f) - \frac{1}{\Sigma_K} \sum_{k \in I_3} \lambda_k \tilde{S}_k(f)
\]

\[
= \sum_{k \in I_1 \cup I_2} \lambda_k \tilde{S}_k(f).
\]
where
\[ \tilde{\Sigma}_K := K - \sum_{k \in \mathcal{I}_2} (1 - \lambda_k) = \sum_{k \in \mathcal{I}_1} 1 + \sum_{k \in \mathcal{I}_2} \lambda_k. \]

Thus, \( \tilde{S}^\text{eig}_K(f) \) and \( S^\text{eig}_K(f) \) can be written as
\[
\tilde{S}^\text{eig}_K(f) = \sum_{k=0}^{N-1} \gamma^\text{eig}_k \tilde{S}_k(f) \quad \text{and} \quad \tilde{S}^\text{eig}_K(f) = \sum_{k=0}^{N-1} \gamma^\text{eig}_k \tilde{S}_k(f)
\]

where
\[
\gamma^\text{eig}_k = \begin{cases} \frac{\lambda_k}{2} & k \in \mathcal{I}_1 \cup \mathcal{I}_2, \\ \lambda_k & k \in \mathcal{I}_3 \cup \mathcal{I}_4, \\ 0 & k \in \mathcal{I}_3. \end{cases}
\]

To bound \( \text{gap}^\text{eig}_k := |\gamma^\text{eig}_k - \tilde{\gamma}^\text{eig}_k| \), we first note that
\[
0 \leq \tilde{\Sigma}_K - \Sigma_K = \sum_{k \in \mathcal{I}_1} (1 - \lambda_k) \leq \epsilon \#(\mathcal{I}_1) \leq K \epsilon,
\]
and
\[
\tilde{\Sigma}_K \geq \Sigma_K = \sum_{k=0}^{K-1} \lambda_k \geq \sum_{k=0}^{K-1} \lambda_{K-1} = K \lambda_{K-1} \geq \frac{K}{2}.
\]

For \( k \in \mathcal{I}_1 \cup \mathcal{I}_2 \), we have \( 0 \leq \lambda_k \leq 1 \), and thus,
\[
\text{gap}^\text{eig}_k = \left| \frac{\lambda_k}{\Sigma_K} - \lambda_k \right| = \frac{\lambda_k (\Sigma_K - \Sigma_K)}{\Sigma_K} = \frac{1}{\Sigma_K} \leq \frac{4 \epsilon}{K}.
\]

For \( k \in \mathcal{I}_3 \) we have \( \gamma^\text{eig}_k = \tilde{\gamma}^\text{eig}_k = 0 \), i.e., \( \text{gap}^\text{eig}_k = 0 \). For \( k \in \mathcal{I}_4 \), we have \( \lambda_k \leq \epsilon \), and thus,
\[
\text{gap}^\text{eig}_k = \left| 0 - \frac{\lambda_k}{\Sigma_K} \right| = \frac{\lambda_k}{\Sigma_K} \leq \frac{\epsilon}{K} = \frac{2 \epsilon}{K} = \frac{4 \epsilon}{K}.
\]

Hence, \( \text{gap}^\text{eig}_k \leq \frac{2 \epsilon}{K} \) for all \( k \in [N] \), and by Lemma 1,
\[
\left| \tilde{S}^\text{eig}_K(f) - \tilde{S}^\text{eig}_K(f) \right| \leq \frac{4 \epsilon}{K} \| x \|^2_2.
\]

Finally, evaluating the approximation
\[
\tilde{S}^\text{eig}_K(f) := \frac{2NW}{\tilde{\Sigma}_K} \tilde{S}^\text{eig}_K(f) - \frac{1}{\Sigma_K} \sum_{k \in \mathcal{I}_3} \lambda_k \tilde{S}_k(f)
\]
at the \( N \) grid frequencies can be done in \( O(N \log^2 N \log \frac{1}{\epsilon}) \) operations in a similar manner as can be done for \( S^\text{eig}_K(f) \).

IV. SIMULATIONS

To test our fast method for multitaper spectral estimation, we first generate \( N = 2^{20} \) samples of an ARMA(12, 8) process. We then try the following methods of spectral estimation:

1) Thomson’s unweighted multitaper method with \( W = 3.6 \times 10^{-5} \) (\( 2NW \approx 75.5 \)), and \( K = 63 \) tapers.

2) Our fast approximation to Thomson’s unweighted multitaper method with \( W = 2.7 \times 10^{-3} \) (\( 2NW \approx 5662.3 \)), \( K = 5641 \) tapers, and an approximation parameter of \( \epsilon = 10^{-12} \).

Note that for both methods, the number of tapers \( K \) was chosen such that \( \lambda_{K-1} > 1 \times 10^{-9} > \lambda_K \), which severely reduces the broadband bias of the tapered estimates. This is necessary due to the high dynamic range of the true spectrum. For the first method, the half-bandwidth parameter \( W = 2.7 \times 10^{-5} \) was chosen according to the optimal number of tapers suggested in [8]. For the second method, the half-bandwidth parameter \( W = 3.6 \times 10^{-5} \) was chosen so that both methods run in a comparable amount of time.

A plot the exact power spectrum of the ARMA(12, 8) process and the errors of the estimated spectra are shown in Figure 1. The precomputation time, run time, and root-mean-squared-logarithmic errors are shown in the table below. Both methods run in approximately the same amount of time due to the fact that our fast approximation only needed to compute \#(\( I_2 \cup I_3 \)) = 56 Slepian tapers. However, the fast approximation has greater accuracy due to the fact that it approximates a multitaper estimate with \( K = 5641 \) tapers.

<table>
<thead>
<tr>
<th>Method</th>
<th>Precomputation time</th>
<th>Time</th>
<th>RMSLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>28.15 s</td>
<td>2.989 s</td>
<td>0.5498 dB</td>
</tr>
<tr>
<td>2</td>
<td>25.33 s</td>
<td>2.932 s</td>
<td>0.1602 dB</td>
</tr>
</tbody>
</table>

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REFERENCES


