Random Gabor Multipliers and Compressive Sensing

Georg Tauböck, Shristi Rajbamshi, Peter Balazs, and Luís Daniel Abreu

Acoustics Research Institute
Austrian Academy of Sciences, 1040 Vienna, Austria
georg.tauboeck@oeaw.ac.at

***Abstract*—We investigate the applicability of Gabor multipliers as compressive measurements. Specifically, we show that the canonical $M \times N$ matrix representation of a Gabor multiplier from $\mathbb{C}^N$ to $\mathbb{C}^M$ is full spark for almost all windows and multiplier symbol vectors with respect to the Lebesgue measure, provided that the length Q of the multiplier symbol vector satisfies $Q \geq N+M-1$. Hence, if used as a measurement matrix in compressive sensing, $M \geq 2S$ guarantees perfect recovery of all S-sparse vectors by means of ℓ_0-minimization for almost all such Gabor multipliers. Furthermore, for ℓ_1-minimization, we demonstrate via simulations that Gabor multipliers with randomly chosen symbol vector have excellent capabilities in recovering sparse signals.**

I. Introduction

Compressive sensing (CS) has been one of the major developments in applied mathematics in the past 15 years [1]–[3]. It considers the recovery of (approximately) sparse vectors from incomplete and possibly perturbed linear measurements via efficient algorithms such as ℓ_1-minimization. Provably optimal bounds for the minimal number of required measurements in terms of the sparsity have been shown for Gaussian and, more generally, subgaussian random matrices [2]–[7]. Practical applications demand for structure in the measurement process which is obviously not present in Gaussian random matrices with independent entries. Several types of structured random matrices have been studied, including random partial Fourier matrices [1], [3], [6], partial random circulant matrices (subsampled random convolutions) [8]–[12], time-frequency structured random matrices [11], [13], [14], and a few more [15], [16].

Clearly, there is large interest in extending CS results to other classes of structured measurements, especially to those relevant for real-world applications. The present contribution aims to make a step towards this direction. In particular, we investigate the applicability of random Gabor multipliers as CS measurements both from a theoretical and practical viewpoint, thereby, extending the random filtering methods of [8] to the time-frequency case. Note that the interest in the consideration of multipliers has its origins in applications. In signal processing many methods employ linear time-invariant filters, i.e., convolution operators, which can be described as Fourier multipliers [17]. Gabor multipliers [18] are a particular option to represent time-varying filters. They have many applications, for example in acoustical signal processing [19], [20] and sparse time-frequency analysis [21].

The work of S.R., P.B., and L.D.A. was supported by the Austrian Science Fund (FWF) START-project Y 551-N13 (FLAME). The work of G.T. and P.B. was supported by the Vienna Science and Technology Fund (WWTF) project MA16-53 (INSIGHT). The work of L.D.A. was supported by the Austrian Science Fund (FWF) project P 31225.

A. Notation

Roman letters $\mathrm{A}, \mathrm{B}, \ldots$, $\mathrm{a}, \mathrm{b}, \ldots$, and $a, b, \ldots$ designate deterministic matrices (operators[1]), vectors, and scalars, respectively. Boldface letters $\mathbf{A}, \mathbf{B}, \ldots$, $\mathbf{a}, \mathbf{b}, \ldots$, and $\mathsf{a}, \mathsf{b}, \ldots$ denote random matrices (operators), random vectors, and random scalars, respectively. The ith component of the vector u (random vector $\mathbf{u}$) is u_{i-1} (u_{i-1}). The superscripts $^\mathsf{T}$ and $^\mathsf{H}$ stand for transposition and Hermitian transposition, respectively. The identity matrix of suitable size is denoted by I. For a vector u, we write supp(u) for its support. For the Euclidean space $(\mathbb{C}^N, \|\cdot\|_2)$, we denote the open ball of radius r centered at $\mathrm{u} \in \mathbb{C}^N$ by $\mathcal{B}_N(\mathrm{u}, r)$. For a set $\mathcal{S}$, $\mathrm{card}(\mathcal{S})$ denotes its cardinality. We use the notation $\mathrm{A}_\mathcal{S}$ to indicate the column submatrix of A consisting of the columns indexed by $\mathcal{S}$. Similarly, for $\mathrm{x} \in \mathbb{C}^N$ we denote by $\mathrm{x}_\mathcal{S}$ the subvector in $\mathbb{C}^{\mathrm{card}(\mathcal{S})}$ consisting of the entries of x indexed by $\mathcal{S}$.

B. Compressive Sensing

In general, reconstruction of $\mathrm{x} = (x_0, \ldots, x_{N-1})^\mathsf{T} \in \mathbb{C}^N$ from measurements

$$\mathrm{y} = \mathrm{A}\mathrm{x}, \tag{1}$$

where $\mathrm{A} \in \mathbb{C}^{M \times N}$ and $M \ll N$, is impossible without substantial *a priori* information about x. In compressive sensing the assumption that x is S-sparse, that is, $\|\mathrm{x}\|_0 \triangleq \mathrm{card}(\mathrm{supp}(\mathrm{x})) \leq S$ for some $S \ll N$, is introduced to guarantee uniqueness and efficient recoverability of x. An estimate $\tilde{\mathrm{x}}$ of the sparse vector x can be obtained by means of solving the ℓ_0-minimization problem,

$$\text{(P0)} \qquad \tilde{\mathrm{x}} = \arg\min_{z} \|\mathrm{z}\|_0 \qquad \text{subject to} \qquad \mathrm{y} = \mathrm{A}\mathrm{z}.$$

This means, we search for the sparsest vector consistent with the measured data $\mathrm{y} = \mathrm{A}\mathrm{x}$. Unfortunately, ℓ_0-minimization is NP-hard in general. Consequently, a number of alternatives,

[1]For notational simplicity, we do not distinguish between operators and their matrix representation with respect to the canonical basis throughout the paper.

for example, greedy algorithms [3], have been proposed in the literature. The most popular approach utilizes ℓ_1–minimization [1], [2], [22], that is, the convex program

$$\text{(P1)} \qquad \tilde{\mathrm{x}} = \arg\min_{\mathrm{z}} \|\mathrm{z}\|_1 \qquad \text{subject to} \qquad \mathrm{y} = \mathrm{Az}$$

is solved, where $\|\mathrm{z}\|_1 = |z_0| + |z_1| + \ldots + |z_{N-1}|$ denotes the usual ℓ_1 vector norm.

Whether the estimate $\tilde{\mathrm{x}}$ obtained by programs (P0) or (P1) coincides with the true solution x in (1), depends on the properties of the so-called measurement matrix A. It is rather elementary to show [3] that $\tilde{\mathrm{x}}$ obtained by (P0) satisfies $\tilde{\mathrm{x}} = \mathrm{x}$ for all S-sparse x, if every set of $2S$ columns of A is linearly independent. To guarantee recoverability of the sparse vector x in (1) by means of program (P1) and greedy algorithms, it suffices to establish the restricted isometry property (RIP) of A [3]. Moreover, the RIP usually implies robustness with respect to noise and ensures that the algorithms produce reliable results not only for perfectly sparse but also for approximately sparse vectors x.

II. Gabor Systems and Gabor Multipliers

A. Gabor Systems

Let us define the following two linear operators on $\mathbb{C}^N$: the *cyclic shift* or *translation* operator $\mathrm{T} : \mathbb{C}^N \to \mathbb{C}^N$, which is given by

$$\mathrm{x} = (x_0, \ldots, x_{N-1})^\mathsf{T} \mapsto \mathrm{Tx} = (x_{N-1}, x_0, \ldots, x_{N-2})^\mathsf{T}$$

and the *modulation* or *frequency shift* operator $\mathrm{M} : \mathbb{C}^N \to \mathbb{C}^N$, which is given by

$$\mathrm{x} = (x_0, \ldots, x_{N-1})^\mathsf{T} \mapsto \mathrm{Mx} = (x_0, \omega x_1, \ldots, \omega^{N-1} x_{N-1})^\mathsf{T},$$

where $\omega = e^{2\pi \mathrm{i}/N}$. The operators $\pi(\lambda) \triangleq \mathrm{M}^l \mathrm{T}^k$, $\lambda = (k, l)$, are called *time-frequency shift* operators. The collection of vectors

$$(\mathrm{g}, \Lambda) \triangleq \{\pi(\lambda)\mathrm{g} : \lambda \in \Lambda\}, \qquad \text{where } \Lambda \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$$

with $\mathbb{Z}_N \triangleq \{0, 1, \ldots, N-1\}$, is denoted as *Gabor system with window* g. If $\Lambda = \mathbb{Z}_N \times \mathbb{Z}_N$, (g, Λ) is called *full* Gabor system with window g. The number $R \triangleq \mathrm{card}(\Lambda)/N$ is referred to as *redundancy* of the Gabor system.

Gabor systems exhibit very rich structural properties. One of these properties (see Theorem 1) will turn out to be instrumental for our investigation of Gabor multipliers in CS. To that end, we need the following definition.

Definition 1. *A family $\mathcal{F}$ of $\mathrm{card}(\mathcal{F}) \geq N$ vectors in $\mathbb{C}^N$ is denoted as* full spark *(cf. [23])—also known as* Haar property *(cf. [24])—if any subset $\mathcal{F}' \subseteq \mathcal{F}$ with $\mathrm{card}(\mathcal{F}') = N$ is linearly independent.*

The following theorem was first proven in [25] for N prime and later generalized to all positive integers $N \in \mathbb{N}$ in [26].

Theorem 1. *For every positive integer N, there is some window $\mathrm{g} \in \mathbb{C}^N$, such that for all $\Lambda \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$ with $\mathrm{card}(\Lambda) \geq N$ the Gabor systems (g, Λ) are full spark. Moreover, the set of such windows g is of full measure, i.e., its complement has N-dimensional Lebesgue measure zero.*

B. Gabor Multipliers

Suppose $N \geq M$ and let $(\mathrm{g}^{(1)}, \Lambda^{(1)})$ and $(\mathrm{g}^{(2)}, \Lambda^{(2)})$ denote two Gabor systems on $\mathbb{C}^N$ and $\mathbb{C}^M$, respectively, i.e., $\mathrm{g}^{(1)} \in \mathbb{C}^N$, $\Lambda^{(1)} \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$, $\mathrm{g}^{(2)} \in \mathbb{C}^M$, and $\Lambda^{(2)} \subseteq \mathbb{Z}_M \times \mathbb{Z}_M$. We further assume that $\mathrm{card}(\Lambda^{(1)}) = \mathrm{card}(\Lambda^{(2)}) \geq N$, which also implies that the redundancies of the two Gabor systems satisfy $R^{(2)} \geq R^{(1)} \geq 1$ as well as that $N \leq M^2$, i.e., N cannot be arbitrarily large for fixed M. Furthermore, let the elements of both sets $\Lambda^{(1)}$ and $\Lambda^{(2)}$ be equipped with some orderings, so that we can write

$$\Lambda^{(1)} = \{\lambda_0^{(1)}, \ldots, \lambda_{Q-1}^{(1)}\} \quad \text{and} \quad \Lambda^{(2)} = \{\lambda_0^{(2)}, \ldots, \lambda_{Q-1}^{(2)}\},$$

where $Q \triangleq \mathrm{card}(\Lambda^{(1)}) = \mathrm{card}(\Lambda^{(2)})$. The orderings are assumed to be arbitrary but fixed.

A *Gabor multiplier* is an operator $\mathrm{G}_{\mathrm{m},\mathrm{g}^{(2)},\mathrm{g}^{(1)}} : \mathbb{C}^N \to \mathbb{C}^M$ defined as[2]

$$\mathrm{x} \;\mapsto\; \mathrm{G}_{\mathrm{m},\mathrm{g}^{(2)},\mathrm{g}^{(1)}}\mathrm{x} \;=\; \sum_{q=0}^{Q-1} m_q \left\langle \mathrm{x}, \pi\big(\lambda_q^{(1)}\big)\mathrm{g}^{(1)} \right\rangle \pi\big(\lambda_q^{(2)}\big)\mathrm{g}^{(2)}$$

with associated *symbol* $\mathrm{m} = (m_0, \ldots, \mathrm{m}_{Q-1})^\mathsf{T} \in \mathbb{C}^Q$.

Remark 1. We note that most definitions of Gabor multipliers restrict to the case $N = M$ and $\Lambda^{(1)} = \Lambda^{(2)}$ with identical orderings, which seems rather natural in order to preserve the time-frequency structure induced by the two Gabor systems. On the other hand, as outlined above, we are interested in using Gabor multipliers (or, more precisely, their matrix representations) as CS measurement matrices, which typically requires $M \ll N$.

Remark 2. As an alternative way[3] of using Gabor multipliers for CS, one could first apply a "conventional" (time-frequency structure preserving) Gabor multiplier $\mathrm{G}_{\mathrm{m},\mathrm{g}^{(2)},\mathrm{g}^{(1)}} : \mathbb{C}^N \to \mathbb{C}^N$ as in Remark 1 and, subsequently, reduce the dimension by subsampling the N-dimensional (time-domain) output vector of the Gabor multiplier. More formally, for a subset $\Omega \subseteq \{0, \ldots, N-1\}$ of cardinality M, let $\mathrm{P}_\Omega : \mathbb{C}^N \to \mathbb{C}^M$, $\mathrm{x} \mapsto \mathrm{x}_\Omega$ be the operator that restricts a vector to the entries indexed by Ω. Then, the overall CS measurement matrix would be expressed as

$$\mathrm{A} = \mathrm{P}_\Omega \mathrm{G}_{\mathrm{m},\mathrm{g}^{(2)},\mathrm{g}^{(1)}} \in \mathbb{C}^{M \times N},$$

and, furthermore,

$$\mathrm{Ax} = \sum_{q=0}^{Q-1} m_q \left\langle \mathrm{x}, \pi\big(\lambda_q\big)\mathrm{g}^{(1)} \right\rangle \mathrm{P}_\Omega \pi\big(\lambda_q\big)\mathrm{g}^{(2)}. \qquad (2)$$

In the following, let us assume regular subsampling, i.e., $M = N/\tau$ with an integer $\tau \geq 2$ and $\Omega = \{\tau m : m = 0, \ldots, M-1\}$. We have, cf. (2),

$$\mathrm{P}_\Omega \pi\big(\lambda_q\big)\mathrm{g}^{(2)} = \mathrm{P}_\Omega \mathrm{M}_N^{l_q} \mathrm{T}_N^{k_q} \mathrm{g}^{(2)} \qquad \text{for some } (k_q, l_q) = \lambda_q,$$

where we have added the subscript $_N$ to modulation operator M and translation operator T, respectively, in order to em-

[2] Such an operator can also be defined for other systems [27].

[3] Based on this approach, random convolutions were investigated as CS measurements and corresponding performances guarantees were established [8]–[12].

phasize their dependence on the dimension. Suppose k_q is a multiple of τ. Then,

$$\begin{aligned}\mathrm{P}_\Omega \pi\big(\lambda_q\big)\mathrm{g}^{(2)} &= \mathrm{M}_M^{l_q}\mathrm{P}_\Omega \mathrm{T}_N^{k_q}\mathrm{g}^{(2)}\\ &= \mathrm{M}_M^{l_q}\mathrm{T}_M^{k_q/\tau}\mathrm{P}_\Omega \mathrm{g}^{(2)}\\ &= \pi\big((k_q/\tau, l_q)\big)\mathrm{g}_\Omega^{(2)}.\end{aligned}$$

Therefore, if $\Lambda^{(1)} = \Lambda^{(2)} = \{(k_0, l_0), \ldots, (k_{Q-1}, l_{Q-1})\}$ is such that k_q is a multiple of τ *for all* $q = 0, \ldots, Q-1$, we can define a new Gabor system $(\tilde{\mathrm{g}}^{(2)}, \tilde{\Lambda}^{(2)})$ with

$$\begin{aligned}&\tilde{\mathrm{g}}^{(2)} \triangleq \mathrm{g}_\Omega^{(2)} \in \mathbb{C}^M \qquad \text{and}\\ &\tilde{\Lambda}^{(2)} \triangleq \{(k_0/\tau, l_0), \ldots, (k_{Q-1}/\tau, l_{Q-1})\},\end{aligned}$$

for which we have

$$\mathrm{A} = \mathrm{P}_\Omega \mathrm{G}_{\mathrm{m},\mathrm{g}^{(2)},\mathrm{g}^{(1)}} = \mathrm{G}_{\mathrm{m},\tilde{\mathrm{g}}^{(2)},\mathrm{g}^{(1)}}.$$

Hence, regular subsampling of a "conventional" Gabor multiplier is included in our general Gabor multiplier definition with $N \geq M$, provided that the "conventional" Gabor system is compatible with the chosen subsampling.

III. Main Results

Consider a Gabor multiplier defined as in Subsection II-B and additionally assume that the windows $\mathrm{g}^{(1)} \in \mathbb{C}^N$ and $\mathrm{g}^{(2)} \in \mathbb{C}^M$ are chosen such that both Gabor systems $(\mathrm{g}^{(1)}, \Lambda^{(1)})$ and $(\mathrm{g}^{(2)}, \Lambda^{(2)})$ are full spark, cf. Definition 1. According to Theorem 1, Lebesgue almost all windows will be suitable. Our main theorem is as follows.

Theorem 2. *Suppose the Gabor systems* $(\mathrm{g}^{(1)}, \Lambda^{(1)})$ *and* $(\mathrm{g}^{(2)}, \Lambda^{(2)})$ *are full spark. Provided that* $Q = \mathrm{card}(\Lambda^{(1)}) = \mathrm{card}(\Lambda^{(2)}) \geq N + M - 1$*, there exists a symbol vector* $\mathrm{m} \in \mathbb{C}^Q$*, such that the family of vectors consisting of the columns of* $\mathrm{G}_{\mathrm{m},\mathrm{g}^{(2)},\mathrm{g}^{(1)}}$ *is full spark. Moreover, the set of such symbol vectors is of full measure, i.e., its complement has* Q*-dimensional Lebesgue measure zero.*

Theorem 2 follows (by means of a union bound argument) from the following Proposition 1, which assumes that the Q-dimensional symbol vector is randomly chosen, thereby, yielding a random $M \times N$ Gabor multiplier matrix $\mathbf{G}_{\mathbf{m},\mathrm{g}^{(2)},\mathrm{g}^{(1)}}$ with a very specific structure/distribution.

Proposition 1. *Suppose the Gabor systems* $(\mathrm{g}^{(1)}, \Lambda^{(1)})$ *and* $(\mathrm{g}^{(2)}, \Lambda^{(2)})$ *are full spark,* $r > 0$*, and* $\mathbf{m}$ *is uniformly distributed on the ball* $\mathcal{B}_Q(0, r)$ *with* $Q \geq N + M - 1$*. Then, the family of vectors consisting of the columns of* $\mathbf{G}_{\mathbf{m},\mathrm{g}^{(2)},\mathrm{g}^{(1)}}$ *is full spark with probability* 1.

The proof of Proposition 1 is somehow involved and cannot be presented due to space limitations. The key ingredient is an important concentration of measure result for random Gabor multipliers (cf. related results in [28]–[31]). Details will be given in an upcoming paper.

Now consider the CS scenario,

$$\mathrm{y} = \mathrm{G}_{\mathrm{m},\mathrm{g}^{(2)},\mathrm{g}^{(1)}}\mathrm{x}, \qquad \|\mathrm{x}\|_0 \leq S, \tag{3}$$

where $\mathrm{G}_{\mathrm{m},\mathrm{g}^{(2)},\mathrm{g}^{(1)}} \in \mathbb{C}^{M\times N}$ is a Gabor multiplier with $N \geq M \geq 2S$ and $Q \geq N + M - 1$. Then, by virtue of Theorem 2, for Lebesgue almost all m, $\mathrm{g}^{(1)}$, and $\mathrm{g}^{(2)}$ (in the sense of Theorem 2), solving the ℓ_0-minimization problem,

$$\text{(P0)} \qquad \tilde{\mathrm{x}} = \arg\min_z \|\mathrm{z}\|_0 \quad \text{subject to} \quad \mathrm{y} = \mathrm{G}_{\mathrm{m},\mathrm{g}^{(2)},\mathrm{g}^{(1)}}\mathrm{z}$$

will yield the true solution $\tilde{\mathrm{x}} = \mathrm{x}$, cf. the discussion in the last paragraph of Subsection I-B. We also would like to emphasize that Theorem 2 implies correct recovery by means of (P0) with probability 1 for all *random* symbol vectors $\mathbf{m}$ whose distribution is absolutely continuous with respect to the Q-dimensional Lebesgue measure.

IV. Numerical Experiments and Discussion

Here, we investigate ℓ_1-minimization to obtain an estimate $\tilde{\mathrm{x}}$ of x for measurements given by (3). Based on our insights about the (P0) program, we choose the multiplier symbol at random and study its recovery performance by means of simulations.

In all our experiments we fix the signal length to $N = 1024$, whereas we vary the number of measurements $M \in \{128, 256, 512\}$. The windows of both Gabor systems, $\mathrm{g}^{(1)} \in \mathbb{C}^N$ and $\mathrm{g}^{(2)} \in \mathbb{C}^M$, are chosen as sampled, periodized Gaussian windows with variance 1. The corresponding time-frequency shift sets $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are assumed to have a rectangular structure, i.e., $\Lambda^{(1)} = \{0, \alpha^{(1)}, 2\alpha^{(1)}, \ldots, (N/\alpha^{(1)} - 1)\alpha^{(1)}\} \times \{0, \beta^{(1)}, 2\beta^{(1)}, \ldots, (N/\beta^{(1)} - 1)\beta^{(1)}\}$ and $\Lambda^{(2)} = \{0, \alpha^{(2)}, 2\alpha^{(2)}, \ldots, (M/\alpha^{(2)} - 1)\alpha^{(2)}\} \times \{0, \beta^{(2)}, 2\beta^{(2)}, \ldots, (M/\beta^{(2)} - 1)\beta^{(2)}\}$, respectively. The exact parameters of the used Gabor systems are specified in Table I.

Gabor system $(\mathrm{g}^{(2)}, \Lambda^{(2)})$	Gabor system $(\mathrm{g}^{(1)}, \Lambda^{(1)})$
$M = 128$ $\alpha^{(2)} = 4$ $\beta^{(2)} = 1$ $R^{(2)} = 32$ $Q = \mathrm{card}(\Lambda^{(2)}) = 4096$	$N = 1024$ $\alpha^{(1)} = 32$ $\beta^{(1)} = 8$ $R^{(1)} = 4$ $Q = \mathrm{card}(\Lambda^{(1)}) = 4096$
$M = 256$ $\alpha^{(2)} = 8$ $\beta^{(2)} = 1$ $R^{(2)} = 32$ $Q = \mathrm{card}(\Lambda^{(2)}) = 8192$	$N = 1024$ $\alpha^{(1)} = 32$ $\beta^{(1)} = 4$ $R^{(1)} = 8$ $Q = \mathrm{card}(\Lambda^{(1)}) = 8192$
$M = 512$ $\alpha^{(2)} = 16$ $\beta^{(2)} = 1$ $R^{(2)} = 32$ $Q = \mathrm{card}(\Lambda^{(2)}) = 16384$	$N = 1024$ $\alpha^{(1)} = 32$ $\beta^{(1)} = 2$ $R^{(1)} = 16$ $Q = \mathrm{card}(\Lambda^{(1)}) = 16384$

TABLE I
Parameters of Gabor systems

For the Q-dimensional symbol vector $\mathbf{m}$ we consider three different random distributions, i.e., (a) $\mathbf{m}$ is uniformly distributed on the ball $\mathcal{B}_Q(0, 1)$, cf. Proposition 1, (b) the components $\mathrm{m}_q \in \mathbb{C}$, $q = 0, \ldots, Q-1$, are i.i.d. and uniformly distributed on the complex unit circle, and (c) $\mathbf{m}$ is a complex standard Gaussian random vector, i.e., $\mathbf{m} \sim \mathcal{CN}(0, \mathrm{I})$. The

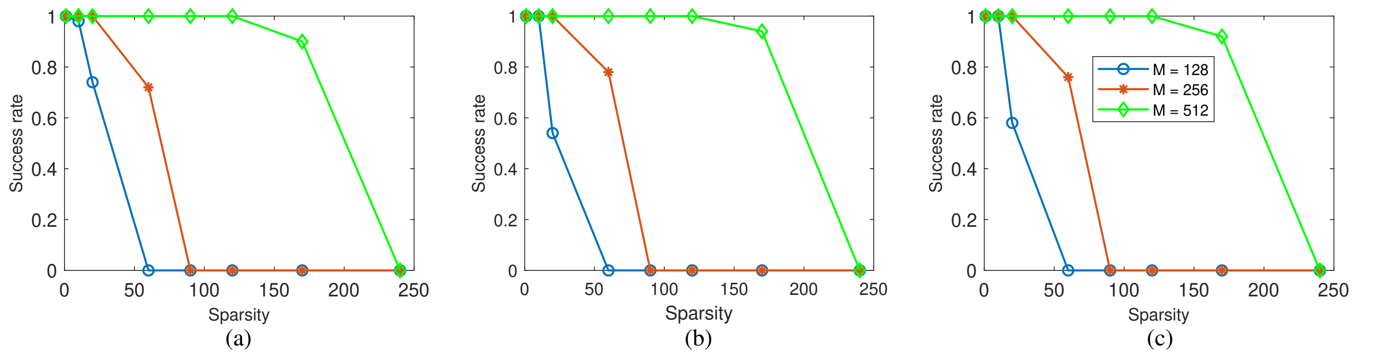

Fig. 1. Success rate of ℓ_1-minimization versus sparsity for different numbers of measurements M. (a) $\mathbf{m}$ uniformly distributed on ball $\mathcal{B}_Q(0,1)$, (b) m_q uniformly distributed on complex unit circle, (c) $\mathbf{m} \sim \mathcal{CN}(0, \mathrm{I})$.

symbol vector is drawn from these distributions once and fixed throughout all further simulations. We study the recovery of S-sparse signals for $S \in \{1, \ldots, 250\}$. To that end, we first generate support sets of cardinality S uniformly at random and, subsequently, draw the nonzero components from the complex standard Gaussian distribution $\mathcal{CN}(0, \mathrm{I})$. Reconstruction is declared as *successful*, if the relative error satisfies $\|\tilde{\mathrm{x}} - \mathrm{x}\|_2 / \|\mathrm{x}\|_2 \leq 10^{-5}$. We repeat this for 50 choices of x for each S. The resulting success rates are depicted in Figure 1. It is clearly seen that random Gabor multipliers show excellent performance in recovering sparse signals via ℓ_1-minimization.

References

[1] E. J. Candès, J. Romberg, and T. Tao, "Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information," *IEEE Trans. Inf. Theory*, vol. 52, no. 2, pp. 489–509, 2006.

[2] D. L. Donoho, "Compressed sensing," *IEEE Trans. Inf. Theory*, vol. 52, no. 4, pp. 1289–1306, Apr. 2006.

[3] S. Foucart and H. Rauhut, *A Mathematical Introduction to Compressive Sensing*, ser. Applied and Numerical Harmonic Analysis. Basel: Birkhäuser, 2013.

[4] S. Mendelson, A. Pajor, and N. Tomczak-Jaegermann, "Reconstruction and subgaussian operators in asymptotic geometric analysis," *Geometric and Functional Analysis*, vol. 17, no. 4, pp. 1248–1282, 2007.

[5] ——, "Uniform uncertainty principle for Bernoulli and subgaussian ensembles," *Constructive Approximation*, vol. 28, no. 3, pp. 277–289, 2008.

[6] M. Rudelson and R. Vershynin, "On sparse reconstruction from Fourier and Gaussian measurements," *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, vol. 61, no. 8, pp. 1025–1045, 2008.

[7] V. Chandrasekaran, B. Recht, P. A. Parrilo, and A. S. Willsky, "The convex geometry of linear inverse problems," *Foundations of Computational Mathematics*, vol. 12, no. 6, pp. 805–849, 2012.

[8] J. A. Tropp, M. B. Wakin, M. F. Duarte, D. Baron, and R. G. Baraniuk, "Random filters for compressive sampling and reconstruction," in *2006 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, vol. 3. IEEE, May 2006, pp. III–III.

[9] J. Romberg, "Compressive sensing by random convolution," *SIAM Journal on Imaging Sciences*, vol. 2, no. 4, pp. 1098–1128, 2009.

[10] H. Rauhut, J. Romberg, and J. A. Tropp, "Restricted isometries for partial random circulant matrices," *Applied and Computational Harmonic Analysis*, vol. 32, no. 2, pp. 242–254, 2012.

[11] F. Krahmer, S. Mendelson, and H. Rauhut, "Suprema of chaos processes and the restricted isometry property," *Communications on Pure and Applied Mathematics*, vol. 67, no. 11, pp. 1877–1904, 2014.

[12] S. Mendelson, H. Rauhut, and R. Ward, "Improved bounds for sparse recovery from subsampled random convolutions," *The Annals of Applied Probability*, vol. 28, no. 6, pp. 3491–3527, 2018.

[13] G. E. Pfander and H. Rauhut, "Sparsity in time-frequency representations," *Journal of Fourier Analysis and Applications*, vol. 16, no. 2, pp. 233–260, 2010.

[14] G. E. Pfander, H. Rauhut, and J. A. Tropp, "The restricted isometry property for time–frequency structured random matrices," *Probability Theory and Related Fields*, vol. 156, no. 3-4, pp. 707–737, 2013.

[15] N. Ailon and H. Rauhut, "Fast and RIP-optimal transforms," *Discrete & Computational Geometry*, vol. 52, no. 4, pp. 780–798, 2014.

[16] M. Hügel, H. Rauhut, and T. Strohmer, "Remote sensing via ℓ_1-minimization," *Foundations of Computational Mathematics*, vol. 14, no. 1, pp. 115–150, Feb 2014.

[17] H. G. Feichtinger and G. Narimani, "Fourier multipliers of classical modulation spaces," *Applied and Computational Harmonic Analysis*, vol. 21, no. 3, pp. 349–359, 2006.

[18] H. G. Feichtinger and K. Nowak, "A first survey of Gabor multipliers," in *Advances in Gabor analysis*. Springer, 2003, pp. 99–128.

[19] P. Balazs, B. Laback, G. Eckel, and W. A. Deutsch, "Time–frequency sparsity by removing perceptually irrelevant components using a simple model of simultaneous masking," *IEEE Transactions on Audio, Speech, and Language Processing*, vol. 18, no. 1, pp. 34–49, 2010.

[20] P. Depalle, R. Kronland-Martinet, and B. Torrésani, "Time-frequency multipliers for sound synthesis," in *Wavelets XII*, vol. 6701. International Society for Optics and Photonics, 2007, pp. 670 118–1–670 118–15.

[21] L. D. Abreu and M. Speckbacher, "A planar large sieve and sparsity of time-frequency representations," in *2017 International Conference on Sampling Theory and Applications (SampTA)*, July 2017, pp. 283–287.

[22] S. S. Chen, D. L. Donoho, and M. A. Saunders, "Atomic decomposition by basis pursuit," *SIAM review*, vol. 43, no. 1, pp. 129–159, 2001.

[23] B. Alexeev, J. Cahill, and D. G. Mixon, "Full spark frames," *Journal of Fourier Analysis and Applications*, vol. 18, no. 6, pp. 1167–1194, Dec. 2012.

[24] E. W. Cheney, *Introduction to Approximation Theory, 2nd ed.* Providence, RI: AMS Chelsea Publishing, 2000.

[25] J. Lawrence, G. E. Pfander, and D. Walnut, "Linear independence of Gabor systems in finite dimensional vector spaces," *Journal of Fourier Analysis and Applications*, vol. 11, no. 6, pp. 715–726, 2005.

[26] R.-D. Malikiosis, "A note on Gabor frames in finite dimensions," *Applied and Computational Harmonic Analysis*, vol. 38, no. 2, pp. 318–330, 2015.

[27] P. Balazs, "Basic definition and properties of Bessel multipliers," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 1, pp. 571–585, 2007.

[28] Y. C. Eldar, D. Needell, and Y. Plan, "Uniqueness conditions for low-rank matrix recovery," *Applied and Computational Harmonic Analysis*, vol. 33, no. 2, pp. 309–314, 2012.

[29] E. Riegler and G. Tauböck, "Almost lossless analog compression without phase information," in *2015 IEEE International Symposium on Information Theory (ISIT)*, Hong Kong, China, June 2015, pp. 999–1003.

[30] G. Tauböck and E. Riegler, "Almost lossless analog compression without phase information - complex case," in *2015 IEEE Information Theory Workshop - Fall (ITW)*, Oct. 2015, pp. 362–366.

[31] D. Stotz, E. Riegler, E. Agustsson, and H. Bölcskei, "Almost lossless analog signal separation and probabilistic uncertainty relations," *IEEE Trans. Inf. Theory*, vol. 63, no. 9, pp. 5445–5460, 2017.