On Identifiability in Unlimited Sampling

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***Abstract*—In recent work [1], the authors introduced the Unlimited Sampling framework which establishes that a bandlimited function can be perfectly recovered from a constant-factor oversampling of its modulo samples, hence complementing recent developments in sensor design. This new sensing framework allows to overcome the clipping or saturation problem that is a fundamental limitation common to all formats of conventional digital sensing that rely on Shannon's sampling theorem. In contrast to critical sampling rate of one sample per second, the sampling density criterion prescribed by the Unlimited Sampling Theorem requires a factor of $2\pi e$ oversampling. In this paper, we prove identifiability conditions linked with the unlimited sensing setup. Our main result establishes that any sampling rate that is faster than critical sampling allows for one-to-one mapping between a finite energy bandlimited function and its modulo samples. This result is corroborated by experiments and opens further interesting questions around the topic as it relaxes the previously held oversampling criterion.**

I. Introduction

Almost all forms of modern data is acquired, stored and transmitted in the digital format. Shannon's sampling theorem [2] is at the heart of all such digital sensing modalities and its impact is so profound that the Third Industrial Revolution is attributed to the digital philosophy. In simple terms, the sampling theorem is a mathematical criterion that asserts that if a continuous function changes slowly, it can be exactly characterized by its discrete samples. How fast (or slow) a function may change is measured using a tool called bandwidth, that is, an upper bound on the fastest frequency of the function. This simple principle allows us to represent the continuum with the discrete.

When it comes to practice, the sampling theorem is implemented using an electronic device whose name aptly justifies its functionality—the analog-to-digital converter or the ADC. A practical problem with such devices is the range of amplitudes that they can handle. In contrast to the sampling theorem which can handle arbitrary range of amplitudes, the ADCs saturate when the input voltage exceeds their recordable limit, say λ. Mathematically, this is a many-to-one mapping where every amplitude value that is larger than the threshold maps to the threshold value. When this happens, the signals are clipped [3]. The clipping or saturation problem is a fundamental bottleneck in digital sensing and several algorithmic approaches have been developed in the literature to mitigate this problem [4]–[9].

A. Unlimited Sensing Scheme

In order to overcome the dynamic range barrier in digital sensing modalities, in our recent work [1], we studied a non-linear sensing methodology that exploits the modulo operation. The modulo an input voltage with respect to maximum recordable voltage λ computes the remainder one dividing the input amplitude by λ. This *many-to-one* mapping has a folding effect, in that, all the amplitudes or voltages $v \in \mathbb{R}$ that are larger than λ map to the range of the ADC, that is, $|v| > \lambda \mapsto [0, \lambda]$. Said differently, $\mathrm{mod}_\lambda(v) \in [0, \lambda]$ and the modulo samples populate the recordable range better than how it would happen for conventional pointwise samples[1]. In practice, modulo operations can be programmed in the hardware and this is indeed the case with self-reset ADCs [10]. However, such ADCs record both the modulo measurements as well as the number of folds. This is different from our approach [1] which only relies on the modulo samples.

In analogy to the Shannon's sampling theorem, the **Unlimited Sampling Theorem (UST)** purely depends on the bandwidth of the function. This is a remarkable feature because one would expect the sampling density to depend on λ, however, this is not the case and hence the name, *unlimited sampling*. The UST essentially trades-off dynamic range with oversampling. For a π–bandlimited function[2], below, we compare the perfect recovery conditions for conventional and unlimited sampling theorems.

	Shannon's Sampling	Unlimited Sampling
Sampling Rate (samples/second)	$\mathrm{T} \leq 1$	$\mathrm{T} \leq \frac{1}{2\pi e}$

In the above, e is the Euler's constant. In the context of UST, what $\mathrm{T} \leq 1/2\pi e$ implies is that oversampled discrete derivatives of the modulo samples are unaffected by the modulo operation. Based on this observation, we proposed a stable recovery algorithm that reconstructs the bandlimited function given modulo samples, in noiseless conditions. This reconstruction proposed in this work was *global* in nature, that is, all modulo samples are required for reconstruction of signals. Following up one our first result, we presented a *local reconstruction scheme* in [11] which allows to unfold a finite subset of samples from modulo samples. This scheme is

[1]This can be verified by comparing the histogram of usual pointwise samples with the modulo samples.

[2]Normalized frequency in radians per second.

particularly useful for parametric signals which are characterized by samples proportional to the number of parameters. For instance, in [11], we discussed recovery of low-pass filtered spikes given modulo samples. In [12], we discussed sampling conditions for sum-of-sinusoids model that is widely studied in spectral estimation theory.

Several papers have followed up on the unlimited sensing framework. In [13], Rudresh and co-workers presented a wavelet based scheme for reconstructing functions acquired using the unlimited sampling method. In working with the noisy scenario, Cucuringu and Tyagi presented an interesting optimization approach for modulo samples in [14]. Compressed sensing of signals within the unlimited sampling architecture was investigated by Musa and co-workers in [15]. Recently, modulo sampling based hardware implementation was discussed in the work of Ordentlich and colleagues [16]. In [17], Graf et al. proposed "one-bit unlimited sampling" with the goal of reconstructing functions from the sign of modulo samples.

B. Contributions

The main goal of this paper is to investigate injectivity conditions for unlimited sampling theory. Our main result shows that $\mathrm{T} < 1$ is sufficient to guarantee a one-to-one mapping between the bandlimited function and its modulo samples. This theoretical result has interesting practical consequences when it comes to designing hardware architectures as ADCs with lower sampling rates can be used for implementing modulo architectures. Furthermore, we provide examples of computer simulations where $\mathrm{T} \in (1/2\pi e, 1]$ allows to unfold modulo samples.

II. Main Results

A. Basic Notation and Useful Definitions

L_2 is the space of square-integrable functions and $\|\cdot\|_{L_2}$ is the associated L_2-norm. Let $\widehat{f}(\omega) = \int f(t) e^{-\jmath\omega t} dt$ denote the Fourier Transform of $f \in L^1$, with its natural extension to arbitrary distributions via duality. We say $f \in C^\infty(\mathbb{R})$ is Ω–bandlimited or,

$$f \in \mathcal{B}_\Omega \Leftrightarrow \quad \widehat{f}(\omega) = \mathbb{1}_{[-\Omega,\Omega]}(\omega) \widehat{f}(\omega)$$

where $\mathbb{1}_{\mathcal{D}}(t)$ is the indicator function on domain $\mathcal{D}$. This includes functions with infinite energy such as a pure sine wave; when restricting to functions of finite energy one obtains the Paley-Wiener space are specified as,

$$\mathsf{PW}_\Omega := f \in \mathcal{B}_\Omega \cap L_2.$$

Note that for $\epsilon > 0$, one has $\mathsf{PW}_\Omega \subset \mathsf{PW}_{\Omega+\epsilon}$.

B. A Useful Result from Complex Analysis

Here we recall a result from complex analysis regarding analytic functions.

Theorem 1. *Let $f(z)$ and $g(z)$ be analytic functions in a common domain $\mathcal{D}$. If $f = g$ on some $\mathcal{D}' \subseteq \mathcal{D}$, $\mathcal{D}'$ having an accumulation point, then $f = g$ everywhere in $\mathcal{D}$.*

Theorem 2. *Let $f_1(z)$ and $f_2(z)$ be analytic functions in a common domain $\mathcal{D}$. If $f_1 = f_2$ on some $\mathcal{D}' \subseteq \mathcal{D}$, $\mathcal{D}'$ having an accumulation point, then $f_1 = f_1$ everywhere in $\mathcal{D}$.*

For further details, we refer to [18] (cf. pg. 124). This result is also known as the **Identity Theorem** in literature.

C. Injectivity Conditions for Unlimited Sampling

Our main result shows that even very small oversampling guarantees unique representation of a signal in PW_π by its modulo samples. To prove this, we begin with the following Lemma.

Lemma 1. *Assume that $f \in \mathsf{PW}_\pi$, $\epsilon > 0$ and $\mathbb{L} \subset \mathbb{Z}$ is some finite set. Then f is uniquely characterized by its samples in $\mathrm{T}_\epsilon \cdot (\mathbb{Z} \setminus \mathbb{L})$ where $0 < \mathrm{T}_\epsilon \leq \frac{\pi}{\pi+\epsilon}$.*

Proof. Assume for a contradiction that there are f, $\epsilon > 0$ and $\mathbb{L}$ that violate the lemma. That is, $\exists g \in \mathsf{PW}_\pi$ such that

$$\forall t \in \mathrm{T}_\epsilon \cdot (\mathbb{Z} \setminus \mathbb{L}), \quad h(t) = (f-g)(t) = 0. \tag{1}$$

Note that as PW_π is a linear space, one has $h \in \mathsf{PW}_\pi \subset \mathsf{PW}_{\pi+\epsilon}$ and the Nyquist rate associated to the latter space corresponds to $\mathrm{T}_\epsilon \leq \frac{\pi}{\pi+\epsilon}$. So we can invoke Shannon's sampling theorem to obtain for an appropriate normalization constant c_ϵ

$$\begin{aligned} h(t) &= c_\epsilon \sum\nolimits_{n\in\mathbb{Z}} h(n\mathrm{T}_\epsilon) \operatorname{sinc}(t/\mathrm{T}_\epsilon - n) \\ &= c_\epsilon \sum\nolimits_{n\in\mathbb{L}} h(n\mathrm{T}_\epsilon) \operatorname{sinc}(t/\mathrm{T}_\epsilon - n) \end{aligned}$$

by assumption. Consequently, in the Fourier domain, the above reads,

$$\widehat{h}(\omega) = \mathrm{T}_\epsilon c_\epsilon \mathbb{1}_{|\omega|<\pi+\varepsilon}(\omega) \underbrace{\sum_{n\in\mathbb{L}} h(n\mathrm{T}_\epsilon) e^{\jmath\omega n\mathrm{T}_\epsilon}}_{\mathrm{H}(\omega)}. \tag{2}$$

Now

$$h \in \mathsf{PW}_\pi \Rightarrow \mathrm{H}(\omega) = 0, \quad \forall\omega \in (\pi, \pi+\epsilon).$$

As $\mathrm{H}(\omega)$ in (2) is a trigonometric polynomial and hence an entire function, Theorem 2 (the Identity Theorem [18]) implies that $\mathrm{H}(\omega)$ and the zero function agree on the full complex plane; consequently $h = 0$ and we obtain a contradiction to our assumption in (1), that is, $f \neq g$. □

Next, we use Lemma 1 to prove that oversampling uniquely determines modulo samples.

Theorem 3 (Injectivity Theorem for Unlimited Sampling)**.** *Any $f \in \mathsf{PW}_\pi$ is uniquely determined by its modulo samples on the grid $\{t_n = n\mathrm{T}_\epsilon\}_{n\in\mathbb{Z}}$ with $\epsilon > 0$.*

Proof. Assume that f is not uniquely determined by its equidistant, pointwise modulo samples, that is, there are $f_1 \neq f_2 \in \mathsf{PW}_\pi$ with

$$f_1(t_n) = \operatorname{mod}_\lambda(f_2(t_n)), \quad t_n = n\mathrm{T}_\epsilon. \tag{3}$$

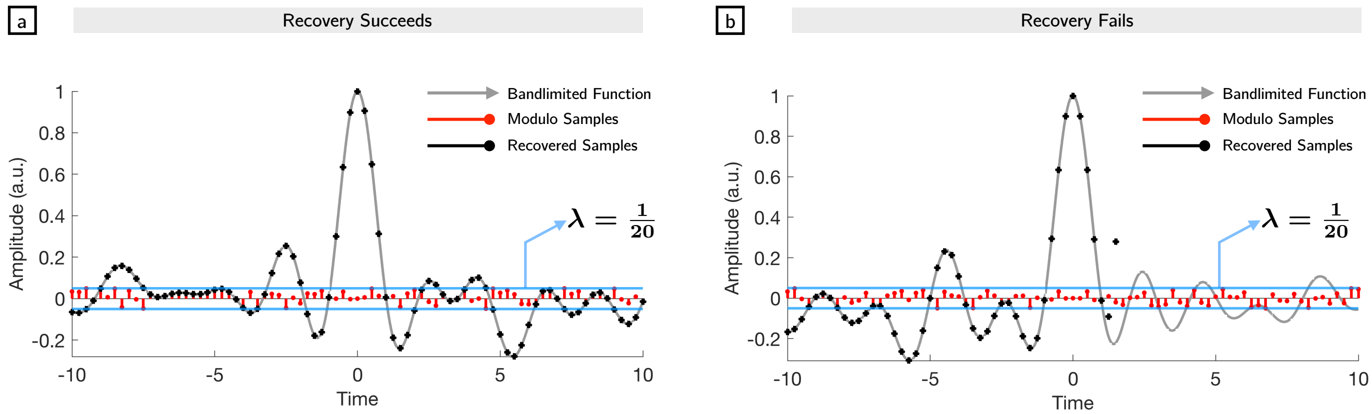

Fig. 1. We compare two realizations of a π–bandlimited signal with absolute maximum value, unity. The maximum recordable voltage is set to be $\lambda = 1/20$ implying that the signal is 20 times larger than the ADC threshold. In either case, we set the sampling rate to be $\mathrm{T} = 1/4$ samples per second. Given modulo samples, we reconstruct the bandlimited signal using the recovery algorithm for unlimited sampling in [1]. (a) Sampling rate $\mathrm{T} = 1/4$ (with 5 orders of forward differences) is enough to recover the signal to its machine precision. This verifies the result of Theorem 3. (b) With the same setup as (a) and even higher orders of differences, recovery is not possible. Repeating the experiment with $\mathrm{T} = 1/2\pi e$ (the sampling rate prescribed by the unlimited sampling theorem), recovery succeeds for both signals.

Since $f_k \in \mathsf{PW}_\pi$, $k = 1, 2$, as before in (1), we have,

$$f_1(t) - f_2(t) \in \mathsf{PW}_\pi \subset L_2.$$

With appropriate c_ϵ, the family of functions $\{c_\epsilon \operatorname{sinc}(t/\mathrm{T}_\epsilon - n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathsf{PW}_{\pi+\epsilon}$, Shannon's sampling theorem yields that,

$$\|f_1(t) - f_2(t)\|_{L_2}^2 = c_\epsilon \sum_{n\in\mathbb{Z}} |f_1(n\mathrm{T}_\epsilon) - f_2(n\mathrm{T}_\epsilon)|^2. \quad (4)$$

We know that each $f_1(n\mathrm{T}_\epsilon) - f_2(n\mathrm{T}_\epsilon) \in \lambda\mathbb{Z}$ as f_1 and f_2 have the same modulo samples. Thus the right hand side of (4) can only be finite if all but finitely many differences $f_1(n\mathrm{T}_\epsilon) - f_2(n\mathrm{T}_\epsilon) \in \lambda\mathbb{Z}$ are zero. Hence we can use Lemma 1 and f_1 and f_2 cannot both be simultaneously in PW_π. $\square$

D. *Experimental Demonstration*

The result of Theorem 3 is experimentally verified in Fig. 1. We generate two bandlimited functions $f \in \mathcal{B}_\pi$ so that the critical sampling rate as per Shannon's sampling theorem and the unlimited sampling theorem is $\mathrm{T}_{\mathsf{Shannon}} = 1$ and $\mathrm{T}_{\mathsf{US}} = 1/2\pi e$, respectively. In order to verify the result of Theorem 3, we set the sampling rate for our experiment to be $\mathrm{T} = 1/4 \in (\mathrm{T}_{\mathsf{US}}, \mathrm{T}_{\mathsf{Shannon}})$. There on, we use the recovery algorithm developed in [1]. As shown in Fig. 1-(a), the method succeeds and the mean squared error (MSE) matches machine precision ($\propto 10^{-33}$). As shown in Fig. 1-(b), under the same settings, the recovery algorithm fails to recover a different realization of $f \in \mathcal{B}_\pi$. Repeating the same experiment again with $\mathrm{T}_{\mathsf{US}} = 1/2\pi e$, both signals are perfectly recovered (again, up to machine precision).

III. Conclusions and Future Directions

Unlimited sampling theorem proves that a bandlimited function can be perfectly recovered from its modulo samples provided that the sampling rate is a factor of $2\pi e$ faster than the critical sampling rate of 1 sample per second. In this paper, we proved that for signals of finite energy any sampling rate faster than critical sampling yields modulo samples that uniquely represent a bandlimited function. This observation suggests that it should be possible in principle that recovery algorithms can be designed that can work with slower sampling rates compared to the $2\pi e$ oversampling. As we have demonstrated in numerical experiments, however, the unlimited sampling setup introduced in [1] requires some modifications before being able to achieve this goal. Another interesting direction to explore would be to which extent the identifiability results derived in this paper carry over to signals that do not have finite energy, such as a combination of pure sine waves.

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