Time-Frequency Shift Invariance of Gabor Spaces

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***Abstract*—We consider non-complete Gabor frame sequences generated by an S_0-function and a lattice Λ and prove that there is $m \in \mathbb{N}$ such that all time-frequency shifts leaving the corresponding Gabor space invariant have their parameters in $\frac{1}{m}\Lambda$. We also investigate time-frequency shift invariance under duality aspects.**

I. Introduction

Due to its many remarkable properties, the Feichtinger Algebra S_0 is known to be a perfect function space for time-frequency analysis. Since functions from S_0 are very well localized in both time and frequency, uncertainty principles in time-frequency analysis (here this is the so-called Amalgam-Balian-Low theorem) prohibit such functions from generating Riesz bases for $L^2(\mathbb{R}^d)$. To be precise, a Gabor system $(g, \Lambda) = \{e^{2\pi ibx} g(x-a) : (a,b) \in \Lambda\}$ with $g \in S_0$ and a lattice $\Lambda \subset \mathbb{R}^{2d}$ can by no means be a Riesz basis for $L^2(\mathbb{R}^d)$.

However, the system (g, Λ) might still constitute a Riesz basis for its closed linear span $\mathcal{G}(g, \Lambda) \neq L^2(\mathbb{R}^d)$. In [3] (see also [4]) the authors observed that – as long as the lattice Λ has rational density – this can only happen if the time-frequency shifts (TFS) that leave $\mathcal{G}(g, \Lambda)$ invariant have parameters in Λ. This generalizes the Amalgam-Balian-Low theorem and was complemented in [5] and [6] by corresponding generalizations of the Balian-Low theorem. However, in all papers [3], [4], [5], [6] it is assumed that the generating lattice has rational density. This is due to the nature of the Zak transform, which is extensively used in the proofs.

In this note we find methods without using the Zak transform and prove two interesting theorems in one dimension ($d = 1$) concerning the TFS-invariance of proper subspaces $\mathcal{G}(g, \Lambda)$ of $L^2(\mathbb{R})$ with S_0-generator g. In Section III we prove (see Theorem III.3) the existence of some $m \in \mathbb{N}$ such that the TFSs under which $\mathcal{G}(g, \Lambda)$ is invariant have parameters in $\frac{1}{m}\Lambda$, if (g, Λ) is a frame for $\mathcal{G}(g, \Lambda)$. This is certainly a strong restriction for such TFSs and is close to the following conjecture: *The TFSs that leave $\mathcal{G}(g, \Lambda)$ invariant have their parameters in Λ if (g, Λ) is a Riesz basis for $\mathcal{G}(g, \Lambda)$*. We mention that our theorem also generalizes the Amalgam-Balian-Low theorem.

In Section IV we characterize the TFS-invariance of $\mathcal{G}(g, \Lambda)$ in terms of the system (g, Λ°), where Λ° is the adjoint lattice corresponding to Λ.

II. Preliminaries

For $a, b \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$ we define the operators of translation by a and modulation by b as

$$T_a f(x) := f(x-a) \quad \text{and} \quad M_b f(x) := e^{2\pi ibx} f(x),$$

respectively. Then both T_a and M_b are unitary operators on $L^2(\mathbb{R})$ and hence so is

$$\pi(a,b) := T_a M_b = e^{-2\pi iab}\, M_b T_a.$$

Let $A \in \mathbb{R}^{n\times n}$ be invertible. Then $\Lambda = A\mathbb{Z}^n$ is called a *lattice* in $\mathbb{R}^n$. The *density* of Λ is defined by $d(\Lambda) = |\det A|^{-1}$. For $g \in L^2(\mathbb{R})$ and a lattice $\Lambda \subset \mathbb{R}^2$ we set

$$(g, \Lambda) := \{\pi(\lambda)g : \lambda \in \Lambda\}$$

and

$$\mathcal{G}(g, \Lambda) := \overline{\operatorname{span}}\,(g, \Lambda).$$

For a closed linear subspace $\mathcal{G} \subset L^2(\mathbb{R})$ let $\mathfrak{I}(\mathcal{G})$ be the set of time-frequency shifts which leave $\mathcal{G}$ invariant, i.e.,

$$\mathfrak{I}(\mathcal{G}) := \{z \in \mathbb{R}^2 : \pi(z)\mathcal{G} \subset \mathcal{G}\}.$$

Lemma II.1 ([2, Prop. A.1])**.** *Let $g \in L^2(\mathbb{R})$, let $\Lambda \subset \mathbb{R}^2$ be a lattice, and set $\mathcal{G} := \mathcal{G}(g, \Lambda)$. Then $z \in \mathfrak{I}(\mathcal{G})$ iff $\pi(z)g \in \mathcal{G}$. Moreover, $\mathfrak{I}(\mathcal{G})$ is a closed additive subgroup of $\mathbb{R}^2$.*

Lemma II.1 shows that $z \in \mathfrak{I}(\mathcal{G})$ implies $-z \in \mathfrak{I}(\mathcal{G})$, i.e., $\pi(z)\mathcal{G} \subset \mathcal{G}$ and $\pi(z)^{-1}\mathcal{G} \subset \mathcal{G}$. Hence, we have in fact that $\pi(z)\mathcal{G} = \mathcal{G}$. Lemma II.1 also implies that for $n \in \mathbb{Z}$ and $\lambda \in \Lambda$,

$$z \in \mathfrak{I}(\mathcal{G}(g, \Lambda)) \implies nz + \lambda \in \mathfrak{I}(\mathcal{G}(g, \Lambda)). \tag{1}$$

In what follows, for a set $\Delta \subset \mathbb{R}^n$ we let $B_\varepsilon(\Delta) := \bigcup_{a\in\Delta} B_\varepsilon(a)$, where $B_\varepsilon(a) := \{x \in \mathbb{R}^n : |x-a| < \varepsilon\}$.

By $S_0(\mathbb{R})$ we denote the *Feichtinger Algebra*, which is the space of functions $f \in L^2(\mathbb{R})$ for which $\langle f, \pi(\cdot)\varphi\rangle \in L^1(\mathbb{R}^2)$ for some (and hence every) Schwarz function $\varphi \neq 0$.

Theorem II.2 ([7, Thm. 13.1.1])**.** *Let $g \in S_0(\mathbb{R})$, $g \neq 0$. Then there exists $\varepsilon > 0$ such that for any lattice $\Lambda \subset \mathbb{R}^2$ with $B_\varepsilon(\Lambda) = \mathbb{R}^2$ the Gabor system (g, Λ) is a frame for $L^2(\mathbb{R})$.*

Corollary II.3. *Let $g \in S_0(\mathbb{R})$, let $\Lambda \subset \mathbb{R}^2$ be a lattice, and set $\mathcal{G} := \mathcal{G}(g, \Lambda)$. Let $\varepsilon > 0$ be as in Theorem* II.2*. If there exist linearly independent vectors $v_1, v_2 \in B_\varepsilon(0) \cap \mathfrak{I}(\mathcal{G})$, then $\mathcal{G} = L^2(\mathbb{R})$.*

Proof. Let A be the matrix with v_1 and v_2 as columns. Then $\Gamma := A\mathbb{Z}^2$ is a lattice in $\mathbb{R}^2$ and, since $v_1, v_2 \in \mathfrak{I}(\mathcal{G})$, we have $\mathcal{G}(g,\Gamma) \subset \mathcal{G}$ by Lemma II.1. Moreover, since also $v_1, v_2 \in B_\varepsilon(0)$, we have $B_\varepsilon(\Gamma) = \mathbb{R}^2$. Using Theorem II.2, we conclude that $L^2(\mathbb{R}) = \mathcal{G}(g,\Gamma) \subset \mathcal{G}$, i.e., $\mathcal{G} = L^2(\mathbb{R})$. □

III. TFS-invariance and Gabor frame sequences

Lemma III.1. *Let $g \in L^2(\mathbb{R})$, let $\Lambda \subset \mathbb{R}^2$ be a lattice, and set $\mathcal{G} := \mathcal{G}(g,\Lambda)$. If there exists $\varepsilon > 0$ such that*

$$B_\varepsilon(\Lambda) \cap \mathfrak{I}(\mathcal{G}) = \Lambda, \tag{2}$$

then there is some $m \in \mathbb{N}$ such that

$$\mathfrak{I}(\mathcal{G}) \subset \tfrac{1}{m}\Lambda.$$

Proof. Let $\Lambda = A\mathbb{Z}^2$ and set $F := A([0,1)^2)$. Then we find sets $C_\ell \subset \mathbb{R}^2$ with diameter $d_\ell < \varepsilon$ such that $F \subset \bigcup_{\ell=1}^{N-1} C_\ell$. Let $z \in \mathfrak{I}(\mathcal{G})$ and consider the N points

$$z_k := kz \mod \Lambda, \qquad k = 0, \dots, N-1,$$

in F. We claim that $z_0, \dots, z_{N-1}$ are not all distinct. Indeed, let us assume towards a contradiction that they are. Then, by the pigeonhole principle, there are $0 \le \ell < k \le N-1$ such that $z_k, z_\ell \in C_j$ for some j and thus $0 < |z_k - z_\ell| < \varepsilon$. But $w_0 := z_k - z_\ell = (k-\ell)z + \lambda$ for some $\lambda \in \Lambda$, and thus $w_0 \in \mathfrak{I}(\mathcal{G})$ by (1); furthermore, $w_0 \in B_\varepsilon(0)$. In fact, we also have $w_0 \notin \Lambda$, since otherwise $z_{k-\ell} = (k-\ell)z \mod \Lambda = 0 = z_0$, contradicting our assumption that $z_0, \dots, z_{N-1}$ are distinct. Thus, $w_0 \in B_\varepsilon(0)\setminus\Lambda$ and $w_0 \in \mathfrak{I}(\mathcal{G})$, which contradicts (2).

Finally, since $z_0, \dots, z_{N-1}$ are not distinct, there are $0 \le \ell < k \le N-1$ with $z_k = z_\ell$, and hence $(k-\ell)z \in \Lambda$. That is, $z \in \frac{1}{n}\Lambda$ for some $n \in \{1, \dots, N-1\}$. Hence, the claim holds for $m := (N-1)!$. □

In the proof of the next lemma we use the technique of metaplectic operators (see [7, Section 9.4]): For any $B \in \mathbb{R}^{2\times 2}$, $\det B = 1$, there exists a unitary operator U_B from $L^2(\mathbb{R})$ onto itself such that $U_B\rho(z) = \rho(Bz)U_B$ for all $z \in \mathbb{R}^2$, where $\rho(a,b) := e^{\pi i ab}\pi(a,b)$.

Lemma III.2. *Let $g \in S_0(\mathbb{R})$ and let $\Lambda \subset \mathbb{R}^2$ be a lattice such that (g,Λ) is a frame for its closed linear span $\mathcal{G} = \mathcal{G}(g,\Lambda)$. If there is a line $L_z = \{tz : t \in \mathbb{R}\}$, $z \in \mathbb{R}^2\setminus\{0\}$, such that $L_z \subset \mathfrak{I}(\mathcal{G})$, then $\mathcal{G} = L^2(\mathbb{R})$.*

Proof. **1.** We first assume that $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$, $\alpha, \beta > 0$, and $L_z = \{(0,t) : t \in \mathbb{R}\}$. Then for $f \in \mathcal{G}$ we have that $M_\omega f \in \mathcal{G}$ for all $\omega \in \mathbb{R}$. By [9, Thm. 9.17], there exists a Borel measurable set $E \subset \mathbb{R}$ such that $\mathcal{G} = L^2(E)$, where we consider $L^2(E)$ as a closed subspace of $L^2(\mathbb{R})$. The goal is to show that $E = \mathbb{R}$ (up to a set of measure zero). To this end, for $f \in L^2(\mathbb{R})$ consider the continuous function $\Gamma_f : \mathbb{R} \to \mathbb{R}$, defined by

$$\Gamma_f(\omega) := \langle SM_\omega f, M_\omega f\rangle, \quad \omega \in \mathbb{R},$$

where S denotes the frame operator of (g,Λ). By [7, Thm. 6.3.2], the operator S has the Walnut representation

$$Sf = \beta^{-1}\sum_{n\in\mathbb{Z}} G_n \cdot T_{\frac{n}{\beta}} f, \quad f \in L^2(\mathbb{R}),$$

with unconditional convergence of the series, and with

$$G_n(x) := \sum_{m\in\mathbb{Z}} g(x - m\alpha)\cdot\overline{g(x - \tfrac{n}{\beta} - m\alpha)}, \qquad x \in \mathbb{R}.$$

Since the multiplication operators G_n and M_ω commute,

$$\Gamma_f(\omega) = \beta^{-1}\sum_{n\in\mathbb{Z}} e^{-2\pi i\frac{n}{\beta}\omega}\langle G_n T_{\frac{n}{\beta}} f, f\rangle.$$

As (g,Λ) is a frame for $\mathcal{G}$ and $M_\omega f \in \mathcal{G}$ for all $\omega \in \mathbb{R}$ and $f \in \mathcal{G}$, there exists $A > 0$ such that $\Gamma_f(\omega) = \langle SM_\omega f, M_\omega f\rangle \ge \beta^{-1}A\|f\|_2^2$ for all $f \in \mathcal{G}$. Integrating this over $[0,\beta]$ gives

$$A\|f\|_2^2 \le \beta^{-1}\sum_{n\in\mathbb{Z}}\langle G_n T_{\frac{n}{\beta}} f, f\rangle \int_0^\beta e^{-2\pi i\frac{n}{\beta}\omega}\,d\omega = \langle hf, f\rangle$$

for all $f \in \mathcal{G} = L^2(E)$, where $h := G_0 = \sum_{m\in\mathbb{Z}}|T_{m\alpha}g|^2$. That is,

$$\int_E (h(x) - A)|f(x)|^2\,dx \ge 0$$

for all $f \in L^2(E)$. Using standard arguments, this implies that $h(x) \ge A$ for a.e. $x \in E$.

Since $T_{m\alpha}g \in \mathcal{G} = L^2(E)$ and thus $T_{m\alpha}g(x) = 0$ for a.e. $x \in \mathbb{R}\setminus E$ and arbitrary $m \in \mathbb{Z}$, it follows that $h(x) = 0$ for a.e. $x \in \mathbb{R}\setminus E$. Note that h is continuous since $g \in S_0$. Hence, the open set $h^{-1}((0,A))$ has zero measure and is thus empty; that is, $h(x) \in \{0\} \cup [A,\infty)$ for all $x \in \mathbb{R}$. By the intermediate value theorem, this implies that $h(x) \ge A$ for all $x \in \mathbb{R}$ (since $h \ge |g|^2$ and $g \not\equiv 0$) and thus, indeed, $E = \mathbb{R}$.

2. Let $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$, $\alpha, \beta > 0$, and $L_z = \{(t,0) : t \in \mathbb{R}\}$. Then $(\widehat{g}, \beta\mathbb{Z} \times \alpha\mathbb{Z})$ is a frame for $\mathcal{F}\mathcal{G}$ and $\{(0,t) : t \in \mathbb{R}\} \subset \mathfrak{I}(\mathcal{F}\mathcal{G})$. Hence, Step 1 yields that $\mathcal{G} = \mathcal{F}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^{-1}L^2(\mathbb{R}) = L^2(\mathbb{R})$.

3. Let $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$, $\alpha, \beta > 0$, and let $L_z = \{tz : t \in \mathbb{R}\}$ be an arbitrary line through the origin, $z \in \mathbb{R}^2\setminus\{0\}$, such that $L_z \subset \mathfrak{I}(\mathcal{G})$. If the line L_z does not hit the lattice Λ (i.e., there is no $t \in \mathbb{R}$ such that $tz \in \Lambda$), then the set $\{tz \mod \Lambda : t \in \mathbb{R}\} \subset \mathfrak{I}(\mathcal{G})$ is dense in $[0,\alpha) \times [0,\beta)$, so that Corollary II.3 yields the claim.

Hence, let us assume that $z = (p\alpha, q\beta)$ for some $p, q \in \mathbb{Z}$. We may assume that $q \ne 0$ since otherwise we are in the situation of Step 2. We may also assume that p and q are co-prime. Let

$$B := \begin{pmatrix} 1 & -(p\alpha)/(q\beta) \\ 0 & 1 \end{pmatrix}$$

and define a new lattice Λ_B by $\Lambda_B := B\Lambda$ and the function $g_B := U_B g \in S_0(\mathbb{R})$ (see [7, Prop. 12.1.3]), where U_B is the metaplectic operator corresponding to B. Then (g_B, Λ_B) is a frame for $\mathcal{G}_B := \mathcal{G}(g_B, \Lambda_B) = U_B\mathcal{G}$. This follows from the fact that

$$\pi(w)g \in \mathcal{G} \implies \pi(Bw)g_B \in U_B\mathcal{G}.$$

The latter also implies that $\pi(t \cdot Bz)g_B \in \mathcal{G}_B$ for $t \in \mathbb{R}$. But $Bz = B(p\alpha, q\beta)^T = (0, q\beta)^T$ so that $\{(0,t) : t \in \mathbb{R}\} \subset \mathfrak{I}(\mathcal{G}_B)$. However, here we are not in the situation of Step 1 because the lattice Λ_B is not separated. It is spanned by the vectors $(\alpha, 0)^T$ and $(-(p/q)\alpha, \beta)$. Since $(0,\beta) \in \mathfrak{I}(\mathcal{G}_B)$, we have that $(kp/q, 0)\alpha \in \mathfrak{I}(\mathcal{G}_B)$ for all $k \in \mathbb{Z}$. Moreover, as $(\ell, 0)\alpha \in \Lambda_B \subset \mathfrak{I}(\mathcal{G}_B)$, it follows that $((kp \mod q)/q, 0)\alpha \in \mathfrak{I}(\mathcal{G}_B)$ for $k \in \mathbb{Z}$. From the fact that p and q are co-prime, we conclude that $(\alpha/q, 0) \in \mathfrak{I}(\mathcal{G}_B)$ and thus $\Gamma := \frac{\alpha}{q}\mathbb{Z} \times \beta\mathbb{Z} \subset \mathfrak{I}(\mathcal{G}_B)$. Now, since $\Gamma = \bigcup_{k=0}^{q-1}(\Lambda_B + (k\alpha/q, 0)^T)$, we obtain

$$(g_B, \Gamma) = \bigcup_{k=0}^{q-1}(g_B, \Lambda_B + (k\alpha/q, 0)^T) = \bigcup_{k=0}^{q-1} T_{-k\alpha/q}\,(g_B, \Lambda_B).$$

And as $T_{-k\alpha/q}\,\mathcal{G}(g_B, \Lambda_B) = \mathcal{G}_B$ for each k, it follows that (g_B, Γ) is a union of q frames for $\mathcal{G}_B$ and therefore itself is a frame for $\mathcal{G}_B$. By Step 1, we find that $\mathcal{G} = U_B^*\mathcal{G}_B = U_B^* L^2(\mathbb{R}) = L^2(\mathbb{R})$.

4. Let Λ and L_z be arbitrary. Then we find a symplectic matrix $C \in \mathbb{R}^{2\times 2}$ such that $\Lambda_C := C\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ with $\alpha, \beta > 0$. Then $(U_C g, \Lambda_C)$ is a frame for $U_C\mathcal{G}$ and $L_{Cz} \subset \mathfrak{I}(U_C\mathcal{G})$. Hence, by the previous steps, $\mathcal{G} = U_C^{-1} L^2(\mathbb{R}) = L^2(\mathbb{R})$ and the claim is proved. □

The following theorem is a consequence of the preceding results.

Theorem III.3. *Let $g \in S_0(\mathbb{R})$ and let $\Lambda \subset \mathbb{R}^2$ be a lattice, such that (g, Λ) is a frame for its closed linear span $\mathcal{G} = \mathcal{G}(g, \Lambda)$. Then exactly one of the following cases applies:*

(a) $\mathfrak{I}(\mathcal{G}) \subset \frac{1}{m}\Lambda$ *for some* $m \in \mathbb{N}\backslash\{0\}$.
(b) (g, Λ) *is a frame for* $L^2(\mathbb{R})$.

Proof. By Lemma III.1, case (a) applies if there exists some $\varepsilon > 0$ such that $B_\varepsilon(\Lambda) \cap \mathfrak{I}(\mathcal{G}) = \Lambda$. If there is no such ε, there exists a sequence $(z_n) \subset \mathfrak{I}(\mathcal{G})\backslash\{0\}$ such that $z_n \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ be as in Theorem II.2 and let $M := \{n \in \mathbb{N} : |z_n| < \varepsilon\}$. Now, either there are $n, m \in M$ such that z_n and z_m are linearly independent, or all z_n, $n \in M$, lie on a line $L_z = \{tz : t \in \mathbb{R}\}$, $z \neq 0$. In the first case we apply Corollary II.3 to get $\mathcal{G} = L^2(\mathbb{R})$, i.e. (b). In the second case, as $\mathfrak{I}(\mathcal{G})$ is closed and closed under addition, it follows that the whole line L_z is contained in $\mathfrak{I}(\mathcal{G})$ and Lemma III.2 implies $\mathcal{G} = L^2(\mathbb{R})$. □

IV. TFS-invariance and Duality

We state the following technical lemma without proof. It shows that in further investigations of TFS-invariance we can restrict ourselves to separable lattices (i.e. $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$) and TFSs in $\mathfrak{I}(\mathcal{G})$ of the form $(\frac{\alpha}{m}, 0)$ with $m \in \mathbb{N}\backslash\{0\}$.

Lemma IV.1. *Let $g \in S_0(\mathbb{R})$ and let $\Lambda \subset \mathbb{R}^2$ be a lattice, such that (g, Λ) is a frame for its closed linear span $\mathcal{G} = \mathcal{G}(g, \Lambda)$. If $\mathfrak{I}(\mathcal{G}) \neq \Lambda$, then there exist $g_1 \in S_0$ and a separable lattice $\Lambda_1 = \alpha\mathbb{Z} \times \beta\mathbb{Z} \subset \mathbb{R}^2$ with $d(\Lambda_1) = d(\Lambda)$ such that (g_1, Λ_1) is a frame for its closed linear span $\mathcal{G}_1 := \mathcal{G}(g_1, \Lambda_1)$ and $(\frac{\alpha}{m}, 0) \in \mathfrak{I}(\mathcal{G}_1)$ for some $m \in \mathbb{N}$, $m > 1$ (i.e., $T_{\frac{\alpha}{m}} g_1 \in \mathcal{G}_1$).*

In what follows let $g \in S_0(\mathbb{R})$, $\alpha, \beta > 0$, $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$, let $\nu \in \mathbb{N}\backslash\{0\}$ be fixed, and assume that (g, Λ) is a frame for $\mathcal{G} = \mathcal{G}(g, \Lambda)$. By γ we denote the canonical dual window (CDW) of g. That is, we have $\langle \pi(\lambda_1)g, \pi(\lambda_2)\gamma\rangle = \delta_{\lambda_1,\lambda_2}$ for $\lambda_1, \lambda_2 \in \Lambda$. Then the adjoint system $\mathcal{F} := \{T_{k/\beta} M_{\ell/\alpha} g : k, \ell \in \mathbb{Z}\}$ is a frame for its closed linear span $\mathcal{K}$ by [8, Thm. 2.2 (c)]. Note that $\mathcal{K} = L^2(\mathbb{R})$ if and only if (g, Λ) is a Riesz sequence (cf.[8, Thm. 2.2 (e)]). Set

$$\mathcal{F}_s := \left\{T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}} M_{\frac{s}{\alpha}} g : k, \ell \in \mathbb{Z}\right\}, \qquad s = 0, \ldots, \nu - 1.$$

Again by [8], $\mathcal{F}_0$ is a frame sequence if and only if the system $(g, \frac{\alpha}{\nu}\mathbb{Z} \times \beta\mathbb{Z})$ is a frame sequence. In this case, each $\mathcal{F}_s$ is a frame sequence because $M_{s/\alpha}\mathcal{F}_0$ is, and multiplying the vectors of a frame sequence by unimodular constants results in a frame sequence. We set $\mathcal{L}_s := \overline{\operatorname{span}}\,\mathcal{F}_s$, $s = 0, \ldots, \nu - 1$. Clearly, we have $\mathcal{K} = \mathcal{L}_0 + \ldots + \mathcal{L}_{\nu-1}$ and $\mathcal{L}_s = M_{\frac{s}{\alpha}}\mathcal{L}_0$.

In the next proposition the (bounded) cross-frame operator $S_{\gamma,g,1/\beta,\nu/\alpha}$ will play a special role. It is defined by

$$S_{\gamma,g} f := S_{\gamma,g,\frac{1}{\beta},\frac{\nu}{\alpha}} f := \sum_{k,\ell\in\mathbb{Z}} \left\langle f, T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}}\gamma\right\rangle T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}} g$$

for $f \in L^2(\mathbb{R})$. Janssen's representation of $S_{\gamma,g}$ (see [7, Ch. 7.2]) is the following:

$$S_{\gamma,g} = \tfrac{\alpha\beta}{\nu} \sum_{m,n\in\mathbb{Z}} \langle g, T_{\frac{m\alpha}{\nu}} M_{n\beta}\gamma\rangle T_{\frac{m\alpha}{\nu}} M_{n\beta}.$$

The series converges absolutely in operator norm: Due to [7, Thm. 7.2.1 and Cor. 12.1.12], to conclude this we only need to verify that $\gamma \in S_0(\mathbb{R})$, which follows from [1, Thm. 7].

In the sequel, the symbol $\dotplus$ denotes the *direct* sum of subspaces. The next theorem characterizes TFS invariance of $\mathcal{G}$ in terms of properties of the adjoint system $\mathcal{F}$.

Theorem IV.2. *Let $g \in S_0(\mathbb{R})$ and assume that $(g, \alpha\mathbb{Z}\times\beta\mathbb{Z})$ is a frame sequence with canonical dual window γ and define the systems $\mathcal{F}_s$ and the spaces $\mathcal{K}, \mathcal{L}_s$ as above, $s = 0, \ldots, \nu - 1$. Then the following statements are equivalent:*

(i) $T_{\frac{\alpha}{\nu}} g \in \mathcal{G}$.
(ii) $(\alpha\beta)^{-1} S_{\gamma,g} M_{\frac{s}{\alpha}} g = \delta_{s0} \cdot g$ *for* $s = 0, \ldots, \nu - 1$.
(iii) $\mathcal{K} = \mathcal{L}_0 \dotplus \ldots \dotplus \mathcal{L}_{\nu-1}$.
(iv) $\left\langle T_{\frac{k}{\beta}} M_{\frac{\ell}{\alpha}}\gamma, g\right\rangle = 0$ *for all* $k, \ell \in \mathbb{Z}$, $\ell \notin \nu\mathbb{Z}$.

If one of (i)–(iv) *holds, then $\mathcal{F}_s$ is a frame for $\mathcal{L}_s$ and the operator*

$$(\alpha\beta)^{-1} M_{s/\alpha} S_{\gamma,g} M_{-s/\alpha}$$

is the (*possibly non-orthogonal*) *projection onto $\mathcal{L}_s$ with respect to the decomposition*

$$L^2(\mathbb{R}) = (\mathcal{L}_0 \dotplus \ldots \dotplus \mathcal{L}_{\nu-1}) \oplus \mathcal{K}^\perp, \quad s = 0, \ldots, \nu - 1.$$

Proof. We will frequently use the following fact (see [8, Thm. 2.3]):

$$(\alpha\beta)^{-1}\gamma \text{ is the CDW of } \mathcal{F} = \{T_{\frac{k}{\beta}} M_{\frac{\ell}{\alpha}} g : k, \ell \in \mathbb{Z}\}. \quad (3)$$

For the rest of the proof we set $P := (\alpha\beta)^{-1} S_{\gamma,g}$. We have $P = \sum_{m,n} c_{mn} T_{\frac{m\alpha}{\nu}} M_{n\beta}$ with $c_{mn} = \frac{1}{\nu}\langle g, T_{\frac{m\alpha}{\nu}} M_{n\beta}\gamma\rangle$. Since $T_{\frac{m\alpha}{\nu}} M_{n\beta}$ commutes with $T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}}$ for all $m, n, k, \ell \in \mathbb{Z}$, it follows immediately that

$$P T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}} = T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}} P. \quad (4)$$

Using (3), we obtain

$$\begin{aligned}
\sum_{s=0}^{\nu-1} M_{\frac{s}{\alpha}} P M_{-\frac{s}{\alpha}} &= \sum_{s=0}^{\nu-1}\sum_{m,n} c_{mn} M_{\frac{s}{\alpha}} T_{\frac{m\alpha}{\nu}} M_{n\beta} M_{-\frac{s}{\alpha}} \\
&= \sum_{m,n} c_{mn}\left(\sum_{s=0}^{\nu-1} e^{2\pi i \frac{ms}{\nu}}\right) T_{\frac{m\alpha}{\nu}} M_{n\beta} \\
&= \nu \sum_{m,n} c_{m\nu,n} T_{m\alpha} M_{n\beta} \qquad (5)\\
&= \sum_{m,n} \langle g, T_{m\alpha} M_{n\beta}\gamma\rangle \cdot T_{m\alpha} M_{n\beta} \\
&= (\alpha\beta)^{-1} S_{\gamma,g,\frac{1}{\beta},\frac{1}{\alpha}} \\
&= S_{(\alpha\beta)^{-1}\gamma,g,\frac{1}{\beta},\frac{1}{\alpha}} = P_{\mathcal{K}},
\end{aligned}$$

where $P_{\mathcal{K}}$ denotes the orthogonal projection onto $\mathcal{K}$. For $s = 0,\dots,\nu-1$ we have $M_{-\frac{s}{\alpha}} P M_{\frac{s}{\alpha}} g =$

$$\begin{aligned}
&= \frac{1}{\nu}\sum_{m,n} \langle g, T_{\frac{m\alpha}{\nu}} M_{n\beta}\gamma\rangle \cdot M_{-\frac{s}{\alpha}} T_{\frac{m\alpha}{\nu}} M_{n\beta} M_{\frac{s}{\alpha}} g \\
&= \frac{1}{\nu}\sum_{m,n}\sum_{r=0}^{\nu-1} \langle g, T_{\frac{\nu m - r}{\nu}\alpha} M_{n\beta}\gamma\rangle M_{-\frac{s}{\alpha}} T_{\frac{\nu m - r}{\nu}\alpha} M_{n\beta} M_{\frac{s}{\alpha}} g \\
&= \frac{1}{\nu}\sum_{r=0}^{\nu-1} e^{2\pi i \frac{sr}{\nu}} T_{-\frac{r\alpha}{\nu}} \sum_{m,n} \langle T_{\frac{r\alpha}{\nu}} g, T_{m\alpha} M_{n\beta}\gamma\rangle T_{m\alpha} M_{n\beta} g \qquad (6)\\
&= \frac{1}{\nu}\sum_{r=0}^{\nu-1} e^{2\pi i \frac{sr}{\nu}} \cdot T_{-\frac{r\alpha}{\nu}} P_{\mathcal{G}} T_{\frac{r\alpha}{\nu}} g,
\end{aligned}$$

where $P_{\mathcal{G}}$ is the orthogonal projection onto $\mathcal{G}$. That is, the vectors

$$v = (M_{-\frac{s}{\alpha}} P M_{\frac{s}{\alpha}} g)_{s=0}^{\nu-1} \quad \text{and} \quad u = (T_{-\frac{r\alpha}{\nu}} P_{\mathcal{G}} T_{\frac{r\alpha}{\nu}} g)_{s=0}^{\nu-1}$$

in $(L^2(\mathbb{R}))^\nu$ are connected via $F_\omega u = \sqrt{\nu}\cdot v$, where F_ω is the DFT-matrix $F_\omega = \nu^{-1/2}(\omega^{sr})_{s,r=0}^{\nu-1}$ with $\omega = e^{2\pi i/\nu}$.

(i)$\Leftrightarrow$(ii). If $T_{\frac{\alpha}{\nu}} g \in \mathcal{G}$, then also $T_{\frac{r\alpha}{\nu}} g \in \mathcal{G}$ for all $r \in \mathbb{Z}$ and (ii) follows from (6). Conversely, if (ii) holds, then $v = (g,0,\dots,0)^T$ and hence $u = \sqrt{\nu}\cdot F_\omega^* v = (g,g,\dots,g)^T$. In particular, $T_{-\alpha/\nu} P_{\mathcal{G}} T_{\alpha/\nu} g = g$, i.e., $T_{\alpha/\nu} g \in \mathcal{G}$.

(ii)$\Rightarrow$(iii). Since $Pg = g$, it is a consequence of (4) that $P|_{\mathcal{L}_0} = I|_{\mathcal{L}_0}$. As for $s \neq 0$ and $k,\ell \in \mathbb{Z}$,

$$P T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}} M_{\frac{s}{\alpha}} g = T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}} P M_{\frac{s}{\alpha}} g = 0,$$

it follows that $P|_{\mathcal{L}_s} = 0$. In general, for $r = 0,\dots,\nu-1$ we have $M_{r/\alpha} P M_{-r/\alpha}|_{\mathcal{L}_r} = I|_{\mathcal{L}_r}$ and $M_{r/\alpha} P M_{-r/\alpha}|_{\mathcal{L}_s} = 0$ for $s \neq r$. Hence, the sum $\mathcal{K} = \mathcal{L}_0 \dotplus \dots \dotplus \mathcal{L}_{\nu-1}$ is direct and $M_{s/\alpha} P M_{-s/\alpha}|_{\mathcal{K}}$ is the projection onto $\mathcal{L}_s$ with respect to this decomposition, $s = 0,\dots,\nu-1$. The fact that $M_{s/\alpha} P M_{-s/\alpha}$ is the projection onto $\mathcal{L}_s$ with respect to the decomposition $L^2(\mathbb{R}) = (\mathcal{L}_0 \dotplus \dots \dotplus \mathcal{L}_{\nu-1}) \oplus \mathcal{K}^\perp$ follows from (5).

(iii)$\Rightarrow$(ii). The relation $P = (\alpha\beta)^{-1} S_{\gamma,g}$ and the definition of $S_{\gamma,g}$ show that $\operatorname{ran} P \subset \mathcal{L}_0$. Hence, obviously $Pg - g \in \mathcal{L}_0$. On the other hand, (5) implies $Pg - g \in \operatorname{span}\{\mathcal{L}_s : s = 1,\dots,\nu-1\}$. Thus, $Pg = g$. Similarly, for $s \in \{1,\dots,\nu-1\}$ we have $M_{s/\alpha} P M_{-s/\alpha} g \in \mathcal{L}_s$, but it also follows from (5) that $M_{s/\alpha} P M_{-s/\alpha} g \in \operatorname{span}\{\mathcal{L}_r : r \neq s\}$. Hence, $P M_{-s/\alpha} g = 0$ for $s = 1,\dots,\nu-1$. Since P commutes with $M_{\pm\nu/\alpha}$ (see (4)), we have $P M_{(\nu-s)/\alpha} g = 0$ and therefore $P M_{s/\alpha} g = 0$ for $s = 1,\dots,\nu-1$.

(i)$\Rightarrow$(iv). Note that also (γ,Λ) is a frame sequence and $\mathcal{G} = \mathcal{G}(\gamma,\Lambda)$. Also, $T_{\alpha/\nu} g \in \mathcal{G}$ iff $\mathcal{G}$ is invariant under $T_{\alpha/\nu}$ iff $T_{\alpha/\nu}\gamma \in \mathcal{G}$. Let us consider the setting above with g and γ interchanged. Define

$$\mathcal{F}_s^* := \left\{ T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}} M_{\frac{s}{\alpha}} \gamma : k,\ell \in \mathbb{Z} \right\}, \qquad s = 0,\dots,\nu-1.$$

Then, cf. (3), $\mathcal{K} = \mathcal{L}_0^* \dotplus \dots \dotplus \mathcal{L}_{\nu-1}^*$, where $\mathcal{L}_s^* := \overline{\operatorname{span}}\,\mathcal{F}_s^*$. We have $S_{g,\gamma} = S_{\gamma,g}^*$. Hence, $M_{s/\alpha} P^* M_{-s/\alpha}$ is the projection onto $\mathcal{L}_s^*$ with respect to the decomposition $L^2(\mathbb{R}) = (\mathcal{L}_0^* \dotplus \dots \dotplus \mathcal{L}_{\nu-1}^*) \oplus \mathcal{K}$. In particular,

$$\mathcal{L}_0^* = \operatorname{ran} P^* = (\ker P)^\perp = (\mathcal{L}_1 \dotplus \dots \dotplus \mathcal{L}_{\nu-1})^\perp \cap \mathcal{K}^\perp.$$

For $k,\ell,m,n \in \mathbb{Z}$ and $s = 1,\dots,\nu-1$ this implies

$$\left\langle T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}} \gamma, T_{\frac{m}{\beta}} M_{\frac{n\nu}{\alpha}} M_{\frac{s}{\alpha}} g \right\rangle = 0,$$

which is equivalent to (iv).

(iv)$\Rightarrow$(ii). Since for $s = 1,\dots,\nu-1$,

$$P M_{\frac{s}{\alpha}} g = (\alpha\beta)^{-1} \sum_{k,\ell\in\mathbb{Z}} \left\langle M_{\frac{s}{\alpha}} g, T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}} \gamma \right\rangle T_{\frac{k}{\beta}} M_{\frac{\ell\nu}{\alpha}} g = 0,$$

$Pg = g$ follows from (5). $\square$

Note that $Pf = f$ for $f \in \mathcal{L}_0$ means that $(\alpha\beta)^{-1}\gamma$ is a dual window for the frame sequence $\mathcal{F}_0 = (g, \frac{1}{\beta}\mathbb{Z} \times \frac{\nu}{\alpha}\mathbb{Z})$. However, it is possible that $\gamma \notin \mathcal{L}_0$.

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