Riesz bases of exponentials for partitions of intervals

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Abstract—**For a partition of** $[0,1]$ **with nodes** $0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1$**, we construct a partition of** $\mathbb{Z}$**,** $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ **such that** $\mathcal{E}(\Lambda_j)$ **is a Riesz basis for** $L^2[a_{j-1}, a_j]$**. Our construction also guarantees that** $\mathcal{E}\left(\cup_{j=1}^k \Lambda_j\right)$ **is a Riesz basis for** $L^2[0, a_k]$**, and** $\mathcal{E}\left(\cup_{j=k+1}^n \Lambda_j\right)$ **is a Riesz basis for** $L^2[a_k, 1]$**.**

I. Introduction

In this paper we develop a method for relating partitions $\{I_n\}_{n=1}^N$ of the interval $[0,1]$ to partitions $\{\Lambda_n\}_{n=1}^N$ of the integers $\mathbb{Z}$ in such a way that the basis properties of the sets

$$\mathcal{E}(\Lambda_n) = \{e^{2\pi i \lambda x} : \lambda \in \Lambda_n\}$$

are preserved under the formation of unions of contiguous intervals in the partition $\{I_n\}$. In particular we prove the following theorem.

Theorem 1.1: Let $0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1$. Then there exists a partition $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_n\}$ of $\mathbb{Z}$ such that

(1) For $j = 1, 2, \ldots, n$, $\mathcal{E}(\Lambda_j)$ is a Riesz basis for $L^2[a_{j-1}, a_j]$.
(2) For every $1 \le k \le n$, $\mathcal{E}\left(\cup_{j=1}^k \Lambda_j\right)$ is a Riesz basis for $L^2[0, a_k]$
(3) For every $0 \le k \le n-1$, $\mathcal{E}\left(\cup_{j=k+1}^n \Lambda_j\right)$ is a Riesz basis for $L^2[a_k, 1]$.

From this we can easily construct Riesz bases of exponentials with integer frequencies to partitions of arbitrary subintervals of $[0, 1]$.

Theorem 1.2: Given $[\alpha, \beta] \subseteq [0, 1]$, and a partition of $[\alpha, \beta]$, $\alpha = a_0 < a_1 < \cdots < a_{n-1} < a_n = \beta$, we construct a collection of disjoint subsets of $\mathbb{Z}$, $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ such that $\mathcal{E}(\Lambda_j)$ is a Riesz basis for $L^2[a_{j-1}, a_j]$ and such that $\mathcal{E}\left(\cup_{j=1}^k \Lambda_j\right)$ is a Riesz basis for $L^2[\alpha, a_k]$, and such that $\mathcal{E}\left(\cup_{j=k+1}^n \Lambda_j\right)$ is a Riesz basis for $L^2[a_k, \beta]$.

We also obtain the following result for infinite partitions of $[0, 1]$.

Theorem 1.3: Let $\{a_k\}_{k \in \mathbb{N}}$ be strictly increasing, $a_1 > 0$ and $a_k \to c$, and let $a_0 = 0$. Then there exists a partition $\{\Lambda_k\}_{k \in \mathbb{N}}$ of $\mathbb{Z}$ with the property that for every $k \in \mathbb{N}$,

(1) $\mathcal{E}(\Lambda_k)$ is a Riesz basis for $L^2[a_{k-1}, a_k]$,
(2) $\mathcal{E}\left(\cup_{j=1}^k \Lambda_j\right)$ is a Riesz basis for $L^2[0, a_k]$ and

(3) $\mathcal{E}\left(\cup_{j=k+1}^\infty \Lambda_j\right)$ is a Riesz basis for $L^2[a_k, 1]$.

Detailed proofs of these results can be found in [8]. In this paper we give outline the proof.

In a recent breakthrough result, Kozma and Nitzan [5] (see also [6]) construct a Riesz basis of the form

$$\mathcal{E}(\Lambda) = \{e^{2\pi i \lambda x} : \lambda \in \Lambda\}$$

where $\Lambda \subseteq \mathbb{Z}$ for

$$L^2(I_1 \cup I_2 \cup \cdots \cup I_n)$$

where $\{I_j\}_{j=1}^n$ is a disjoint collection of subintervals of $[0, 1]$. As a central part of the proof, the authors show that under some circumstances it is possible to "combine Riesz bases" in the sense described above, but as the authors point out in the paper, their construction does not necessarily give subsets Λ_j such that $\mathcal{E}(\Lambda_j)$ forms a Riesz basis for $L^2(I_j)$ and $\Lambda = \cup_{j=1}^n \Lambda_j$. Indeed it is in general not true that the union of Riesz bases for disjoint intervals forms a Riesz basis for the union of the intervals (see [5]).

II. One interval and Seip's Lemma

Theorem 1.2 for $n = 1$ follows immediately from Lemma 2.1 of Seip by the translation invariance property of exponential bases.

Lemma 2.1: (Seip [10]) For any $0 < \alpha < 1$, there exists a subset $\Lambda \in \mathbb{Z}$ such that $\mathcal{E}(\Lambda)$ is a Riesz basis for $L^2[0, \alpha]$.

The main tool in the proof of this lemma is Avdonin's Theorem, a version of which is given below.

Theorem 2.2: (Avdonin [1]) Let $L > 0$ and suppose that there exists an injective map

$$\varphi \colon \frac{1}{L}\mathbb{Z} \to \mathbb{R}$$

with the property that the range of φ is separated[1] and such that for some $c, R > 0$

$$\sup_{m \in \mathbb{Z}} \left| \sum_{\frac{k}{L} \in [mR, (m+1)R)} \varphi\left(\frac{k}{L}\right) - \left(\frac{k}{L} - c\right) \right| < \frac{1}{4L}. \quad (1)$$

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[1]A set $S \subseteq \mathbb{R}$ is *separated* if there exists $\delta > 0$ such that if $x, y \in S$, $x \neq y$ then $|x - y| \ge \delta$.

Then $\mathcal{E}\big(\varphi\big(\frac{1}{L}\,\mathbb{Z}\big)\big)$ is a Riesz basis for $L^2(I)$ for any interval I with $|I| = L$. We refer to the quantity on the left side of (1) as the *Avdonin bound* of $\varphi\big(\frac{k}{L}\big)$ with respect to $\frac{1}{L}\mathbb{Z}$.

The proof of Lemma 2.1 is in the same spirit as in [10] but differs in such a way as to allow for extensions like Theorem 3.2 for two intervals and provides a theme that reappears in the proof of Theorem 1.1 and its variants for n intervals.

Proof: (Lemma 2.1, [9]) Let $\epsilon > 0$ and define $\varphi\colon \frac{1}{\alpha}\mathbb{Z} \to \mathbb{Z}$ to be the function that rounds each $\frac{n}{\alpha}$ to the nearest integer, i.e.,

$$\varphi\left(\frac{n}{\alpha}\right) = \begin{cases} \left\lfloor \frac{n}{\alpha} \right\rfloor & \text{if } \left[\frac{n}{\alpha}\right] < 1/2 \\[2mm] \left\lceil \frac{n}{\alpha} \right\rceil & \text{if } \left[\frac{n}{\alpha}\right] > 1/2 \end{cases} \tag{2}$$

with an appropriate definition being made when $\frac{n}{\alpha}$ is a half-integer. Here and in what follows, $[x] = x \pmod 1$, $\lfloor x \rfloor$ is the largest integer less than or equal to x, and $\lceil x \rceil$ is the smallest integer greater than or equal to x.

Next we want to show that there exists $N \in \mathbb{N}$ such that

$$\left| \frac{1}{N} \sum_{n=mN}^{(m+1)N-1} \varphi\left(\frac{n}{\alpha}\right) - \frac{n}{\alpha} \right| < \epsilon. \tag{3}$$

To see that this is the case, suppose first that α is rational with $\alpha = \frac{N_0}{K_0}$ in lowest terms. Clearly, $\frac{1}{\alpha}\mathbb{Z}$ has period K_0 (that is, $\frac{1}{\alpha}\mathbb{Z} + K_0 = \frac{1}{\alpha}\mathbb{Z}$) and so it suffices to consider the sum in (3) only for $m = 0$ and N a multiple of K_0. It can be shown that if $N = 2K_0$, and with an appropriate choice of $\varphi\big(\frac{n}{\alpha}\big)$ when $\frac{n}{\alpha}$ is a half-integer, that the sum in (3) vanishes identically.

If α is irrational, define for $x \in [0,1]$,

$$f(x) = \begin{cases} -x & \text{if } x \in [0, 1/2] \\ 1-x & \text{if } x \in (1/2, 1] \end{cases}$$

and note that $\varphi(x) - x = f([x])$. Since f is piecewise continuous, it can be shown using a variation of the proof of the Weyl Equidistribution Theorem that given $\epsilon > 0$, then for all $N \in \mathbb{N}$ sufficiently large,

$$\left| \frac{1}{N} \sum_{n=mN}^{(m+1)N} f\left(\left[\frac{n}{\alpha}\right]\right) - \int_0^1 f(t)\,dt \right| = \left| \frac{1}{N} \sum_{n=mN}^{(m+1)N} f\left(\left[\frac{n}{\alpha}\right]\right) \right| < \epsilon$$

and (3) follows. Finally we note that by choosing $\epsilon < \frac{1}{4\alpha}$, letting $\Lambda = \varphi\big(\frac{1}{\alpha}\mathbb{Z}\big)$, and applying Theorem 2.2, the lemma follows. ∎

III. TWO INTERVALS

From general principles it can be shown that, with α and Λ as in Lemma 2.1 $\mathcal{E}(\mathbb{Z} \setminus \Lambda)$ is a Riesz basis for $L^2[\alpha, 1]$, which amounts to a proof of Theorem 1.1 for two intervals. The result follows from a theorem related to the notion of the Naimark's complement of an orthogonal basis (see [7], [2], [3]). For our purpose, we state a special case of that theorem.

Theorem 3.1: Let $0 < \alpha < 1$, $\mathcal{H} = L^2[0,1]$ and $P\colon L^2[0,1] \to L^2[0,\alpha]$ be given by $Pf = f\chi_{[0,\alpha]}$. Suppose that we partition $\mathbb{Z}$ as $\{\Lambda_1, \Lambda_2\}$ and that $\mathcal{E}(\Lambda_1)$ is a Riesz basis for $L^2[0,\alpha]$. Then $\mathcal{E}(\Lambda_2)$ is a Riesz basis for $L^2[\alpha, 1]$.

In order to extend Theorem 1.1 to more than two intervals, we make the further surprising observation (see [9]) that the set $\mathbb{Z} \setminus \Lambda$ is obtained by rounding each point of the set $\frac{1}{1-\alpha}\mathbb{Z} + \frac{1}{2(1-\alpha)}$ to its nearest integer (with an appropriate choice being made when such point is a half-integer). Theorem 3.2 forms an extension of Lemma 2.1 and when combined with Theorem 2.2, gives Theorem 1.1 when $n = 2$. We present some details in this case as it forms the framework for the general result for arbitrary n.

Theorem 3.2: [9] Given $0 < \alpha < 1$ and $\epsilon > 0$, define φ on $\frac{1}{\alpha}\mathbb{Z}$ by (2) and ψ on $\frac{1}{1-\alpha}\mathbb{Z}$ by

$$\psi\left(\frac{n}{1-\alpha}\right)$$
$$= \begin{cases} \left\lfloor \frac{n}{1-\alpha} + \frac{1}{2(1-\alpha)} \right\rfloor & \text{if } \left[\frac{n}{1-\alpha} + \frac{1}{2(1-\alpha)}\right] < 1/2 \\[3mm] \left\lceil \frac{n}{1-\alpha} + \frac{1}{2(1-\alpha)} \right\rceil & \text{if } \left[\frac{n}{1-\alpha} + \frac{1}{2(1-\alpha)}\right] > 1/2 \end{cases} \tag{4}$$

where φ and ψ are defined appropriately on half-integers. Then
(1) $\big\{\varphi\big(\frac{1}{\alpha}\mathbb{Z}\big), \psi\big(\frac{1}{1-\alpha}\mathbb{Z}\big)\big\}$ forms a partition of $\mathbb{Z}$, and
(2) there exists $N \in \mathbb{N}$ such that with $R = \frac{N}{1-\alpha}$,

$$\sup_{m \in \mathbb{Z}} \left| \frac{1}{R} \sum_{\substack{\frac{j}{\alpha} \in [mR, (m+1)R)}} \varphi\left(\frac{j}{\alpha}\right) - \frac{j}{\alpha} \right| < \epsilon, \tag{5}$$

and

$$\sup_{m \in \mathbb{Z}} \left| \frac{1}{R} \sum_{\substack{\frac{j}{1-\alpha} + \frac{1}{2(1-\alpha)} \\ \in [mR, (m+1)R)}} \psi\left(\frac{j}{1-\alpha}\right) - \left(\frac{j}{1-\alpha} + \frac{1}{2(1-\alpha)}\right) \right| < \epsilon. \tag{6}$$

Remark 3.3: The ambiguity in the definitions of φ and ψ when $\frac{n}{\alpha}$ and $\frac{k}{1-\alpha} + \frac{1}{2(1-\alpha)}$ are half-integers will only occur when α is rational and, if $\alpha = \frac{N_0}{K_0}$ in lowest terms, when N_0 is even. In this case, when $\frac{n}{\alpha}$ is a half integer, $\frac{n}{\alpha} = \frac{k}{1-\alpha} + \frac{1}{2(1-\alpha)}$ for some $k \in \mathbb{Z}$, and we define $\varphi\big(\frac{n}{\alpha}\big) = \big\lfloor \frac{n}{\alpha} \big\rfloor$ and $\psi\big(\frac{k}{1-\alpha}\big) = \big\lceil \frac{k}{\alpha} + \frac{1}{2(1-\alpha)} \big\rceil$ or vice versa so that the points are mapped to distinct integers. This guarantees that the sums in (5) and (6) vanish identically.

The fundamental observations leading to the proof of Theorem 3.2 are contained in the following lemma.

Lemma 3.4: Given $j \in \mathbb{Z}$, let $I = \big[\frac{j-1}{1-\alpha}, \frac{j}{1-\alpha}\big)$. Then the following hold.
(1) If there are exactly K elements of $\frac{1}{\alpha}\mathbb{Z}$ in I, then there are exactly $K + 1$ integers in I.
(2) If $\frac{n}{\alpha} \in I$, and $\frac{n}{\alpha} < \frac{j-1}{1-\alpha} + \frac{1}{2(1-\alpha)}$ then $\varphi\big(\frac{n}{\alpha}\big) = \big\lfloor \frac{n}{\alpha} \big\rfloor$ and if $\frac{n}{\alpha} > \frac{j-1}{1-\alpha} + \frac{1}{2(1-\alpha)}$ then $\varphi\big(\frac{n}{\alpha}\big) = \big\lceil \frac{n}{\alpha} \big\rceil$.
(3) The numbers $\frac{n}{\alpha}$ and $\frac{k}{1-\alpha} + \frac{1}{2(1-\alpha)}$ are never within distance $\frac{1}{2}$ of the same integer, unless they are equal.
(4) If for some $k \in \mathbb{Z}$, $\frac{n}{\alpha} = \frac{k}{1-\alpha} + \frac{1}{2(1-\alpha)}$, then the common value is a half-integer.

Proof: (Theorem 3.2) By considering intervals of the form $I = \big[\frac{j-1}{1-\alpha}, \frac{j}{1-\alpha}\big)$, for $j \in \mathbb{Z}$, we first observe that by Lemma 3.4(3) and (4), $\varphi\big(\frac{n}{\alpha}\big)$ and $\psi\big(\frac{j-1}{1-\alpha} + \frac{1}{2(1-\alpha)}\big)$ will never map to the same integer, so that the ranges of φ and ψ restricted to I are disjoint. Also, by Lemma 3.4(1), all integers

in I are in the image of either φ or ψ restricted to I. From this (1) follows.

To see that (2) holds, we observe that, as in the proof of Lemma 2.1, if α is rational with $\alpha = \frac{N_0}{K_0}$ in lowest terms, then the sets $\frac{1}{\alpha}\mathbb{Z}$, $\frac{1}{1-\alpha}\mathbb{Z} + \frac{1}{2(1-\alpha)}$, and $\mathbb{Z}$ have a common period K_0 and that for $k \in [0, 2K_0)$, the sums (5) and (6) vanish identically with an appropriate choice of φ and ψ on the half-integers.

If α is irrational, then $\psi(x) - (x - \frac{1}{2(1-\alpha)}) = f([x - \frac{1}{2(1-\alpha)}])$ where f is defined in the proof of Lemma 2.1. Using the same equidistribution argument, (6) follows. ∎

A simple dilation argument gives the following corollary.

Corollary 3.5: Let $0 < a < b$, and $\epsilon > 0$. Then there exist injective maps

$$\varphi\colon \frac{1}{a}\mathbb{Z} \longrightarrow \frac{1}{a+b}\mathbb{Z}, \qquad \psi\colon \frac{1}{b}\mathbb{Z} \longrightarrow \frac{1}{a+b}\mathbb{Z}$$

such that

(1) $\{\varphi(\frac{1}{a}\mathbb{Z}), \psi(\frac{1}{b}\mathbb{Z})\}$ forms a partition of $\frac{1}{a+b}\mathbb{Z}$, and
(2)
$$\sup_{m\in\mathbb{Z}} \left| \frac{1}{R} \sum_{\frac{j}{a}\in[mR,(m+1)R)} \varphi\left(\frac{j}{a}\right) - \frac{j}{a} \right| < \epsilon, \qquad (7)$$

and

$$\sup_{m\in\mathbb{Z}} \left| \frac{1}{R} \sum_{\substack{\frac{j}{b}+\frac{1}{2b} \\ \in[mR,(m+1)R)}} \psi\left(\frac{j}{b}\right) - \left(\frac{j}{b}+\frac{1}{2b}\right) \right| < \epsilon. \qquad (8)$$

IV. THREE INTERVALS

In this section we will describe how to prove Theorem 1.1 for a partition of $[0,1]$ into three intervals. This will illustrate the fundamental idea that leads to the general result.

Suppose that $0 = a_0 < a_1 < a_2 < a_3 = 1$. Applying Theorem 3.2 with $\alpha = a_2$ and some $\epsilon > 0$ we can define injections

$$\Psi\colon \frac{1}{1-a_2}\mathbb{Z} \to \mathbb{Z}, \qquad \sigma\colon \frac{1}{a_2}\mathbb{Z} \to \mathbb{Z}$$

such that (1) and (2) hold. If $0 < \epsilon < \frac{1}{4}$, we may apply Theorem 2.2 and assert that $\mathcal{E}\big(\Psi\big(\frac{1}{1-a_2}\mathbb{Z}\big)\big)$ is a Riesz basis for $L^2[a_2, 1]$, and that $\mathcal{E}\big(\sigma\big(\frac{1}{a_2}\mathbb{Z}\big)\big)$ is a Riesz basis for $L^2[0, a_2]$.

Now applying Corollary 3.5 with $a = a_1$ and $b = a_2 - a_1$, so that $a+b = a_2$ and choosing $\delta > 0$, we can define injections

$$\widehat{\varphi}\colon \frac{1}{a_1}\mathbb{Z} \to \frac{1}{a_2}\mathbb{Z}, \qquad \widehat{\psi}\colon \frac{1}{a_2-a_1}\mathbb{Z} \to \frac{1}{a_2}\mathbb{Z}$$

such that Corollary 3.5(1) and (2) hold with δ replacing ϵ. Now consider the following diagram.

$$\begin{array}{ccccc} \frac{1}{a_1}\mathbb{Z} & \xrightarrow{\widehat{\varphi}} & \frac{1}{a_2}\mathbb{Z} & \xrightarrow{\sigma} & \mathbb{Z} \\ \frac{1}{a_2-a_1}\mathbb{Z} & \xrightarrow{\widehat{\psi}} & \frac{1}{a_2}\mathbb{Z} & \xrightarrow{\sigma} & \mathbb{Z}. \end{array}$$

Note that because $\{\widehat{\varphi}\big(\frac{1}{a_1}\mathbb{Z}\big), \widehat{\psi}\big(\frac{1}{a_2-a_1}\mathbb{Z}\big)\}$ forms a partition of $\frac{1}{a_2}\mathbb{Z}$, $\{\sigma \circ \widehat{\varphi}\big(\frac{1}{a_1}\mathbb{Z}\big), \sigma \circ \widehat{\psi}\big(\frac{1}{a_2-a_1}\mathbb{Z}\big)\}$ forms a partition of

$\sigma\big(\frac{1}{a_2}\mathbb{Z}\big)$. Also, since $\{\Psi\big(\frac{1}{1-a_2}\mathbb{Z}\big), \sigma\big(\frac{1}{a_2}\mathbb{Z}\big)\}$ forms a partition of $\mathbb{Z}$, $\{\sigma \circ \widehat{\varphi}\big(\frac{1}{a_1}\mathbb{Z}\big), \sigma \circ \widehat{\psi}\big(\frac{1}{a_2-a_1}\mathbb{Z}\big), \Psi\big(\frac{1}{1-a_2}\mathbb{Z}\big)\}$ also forms a partition of $\mathbb{Z}$.

We wish now to assert that for some $R > 0$,

$$\sup_{m\in\mathbb{Z}} \left| \frac{1}{R} \sum_{\frac{k}{a_1}\in[mR,(m+1)R)} \sigma \circ \widehat{\varphi}\left(\frac{k}{a_1}\right) - \frac{k}{a_1} \right|$$

and

$$\sup_{m\in\mathbb{Z}} \left| \frac{1}{R} \sum_{\substack{\frac{k}{a_2-a_1}+\frac{1}{2(a_2-a_1)} \\ \in[mR,(m+1)R)}} \sigma \circ \widehat{\psi}\left(\frac{k}{a_2-a_1}\right) - \left(\frac{k}{a_2-a_1} - \frac{1}{2(a_2-a_1)}\right) \right|$$

are sufficiently small that Theorem 2.2 can be applied so that we can then assert that $\mathcal{E}\big(\sigma \circ \widehat{\varphi}\big(\frac{1}{a_1}\mathbb{Z}\big)\big)$ is a Riesz basis for $L^2[0, a_1]$, and that $\mathcal{E}\big(\sigma \circ \widehat{\psi}\big(\frac{1}{a_2-a_1}\mathbb{Z}\big)\big)$ is a Riesz basis for $L^2[a_1, a_2]$. However, in order to guarantee that this is the case, we must locally modify the injections $\widehat{\varphi}$ and $\widehat{\psi}$ so that the above averages are small.[2] This is done via the following theorem which forms the crux of the general argument. The proof is quite intricate and can be found in [8].

Theorem 4.1: Suppose that for some $a, b, \epsilon, \delta > 0$ there exist injective maps

$$\widehat{\varphi}\colon \frac{1}{a}\mathbb{Z} \to \frac{1}{a+b}\mathbb{Z}, \quad \widehat{\psi}\colon \frac{1}{b}\mathbb{Z} \to \frac{1}{a+b}\mathbb{Z}, \quad \sigma\colon \frac{1}{a+b}\mathbb{Z} \to \mathbb{Z}$$

such that

(1) $\{\widehat{\varphi}\big(\frac{1}{a}\mathbb{Z}\big), \widehat{\psi}\big(\frac{1}{b}\mathbb{Z}\big)\}$ forms a partition of $\frac{1}{a+b}\mathbb{Z}$,
(2) there exists $\widehat{M} > 0$ such that for all $j \in \mathbb{Z}$, $\big|\widehat{\varphi}\big(\frac{j}{a}\big) - \frac{j}{a}\big|$, $\big|\widehat{\psi}\big(\frac{j}{b}\big) - \big(\frac{j}{b} - \frac{1}{2b}\big)\big|$ and $\big|\sigma\big(\frac{j}{a+b}\big) - \frac{j}{a+b}\big|$ are bounded by $\widehat{M}$, and
(3) there exists $\widehat{R} > 0$ such that

$$\sup_{m\in\mathbb{Z}} \left| \frac{1}{\widehat{R}} \sum_{\frac{k}{a}\in[m\widehat{R},(m+1)\widehat{R})} \widehat{\varphi}\left(\frac{k}{a}\right) - \frac{k}{a} \right| < \delta,$$

$$\sup_{m\in\mathbb{Z}} \left| \frac{1}{\widehat{R}} \sum_{\substack{\frac{k}{b}+\frac{1}{2b} \\ \in[m\widehat{R},(m+1)\widehat{R})}} \widehat{\psi}\left(\frac{k}{b}\right) - \left(\frac{k}{b}+\frac{1}{2b}\right) \right| < \delta,$$

and

$$\sup_{m\in\mathbb{Z}} \left| \frac{1}{\widehat{R}} \sum_{\frac{k}{a+b}\in[m\widehat{R},(m+1)\widehat{R})} \sigma\left(\frac{k}{a+b}\right) - \frac{k}{a+b} \right| < \epsilon.$$

Then there exist injective maps

$$\varphi\colon \frac{1}{a}\mathbb{Z} \to \frac{1}{a+b}\mathbb{Z}, \qquad \psi\colon \frac{1}{b}\mathbb{Z} \to \frac{1}{a+b}\mathbb{Z}$$

and $R > 0$ such that

(4) $\{\varphi\big(\frac{1}{a}\mathbb{Z}\big), \psi\big(\frac{1}{b}\mathbb{Z}\big)\}$ forms a partition of $\frac{1}{a+b}\mathbb{Z}$,
(5) for all $j \in \mathbb{Z}$, $\big|\sigma \circ \varphi\big(\frac{j}{a}\big) - \frac{j}{a}\big|$ and $\big|\sigma \circ \psi\big(\frac{j}{b}\big) - \big(\frac{j}{b} - \frac{1}{2b}\big)\big|$ are bounded by $\widehat{M} + R$, and

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[2]For the case of three intervals, it is shown in [9] using a different argument that no modification of $\widehat{\varphi}$ and $\widehat{\psi}$ is required. However, Theorem 4.1 is required for $n > 3$.

(6) for all $m \in \mathbb{Z}$,

$$\left| \frac{1}{R} \sum_{\frac{k}{b} \in [mR,(m+1)R)} \sigma \circ \varphi\left(\frac{k}{a}\right) - \frac{k}{a} \right| < \epsilon + 3\delta,$$

$$\left| \frac{1}{R} \sum_{\substack{\frac{k}{b} + \frac{1}{2b} \\ \in [mR,(m+1)R)}} \sigma \circ \psi\left(\frac{k}{b}\right) - \left(\frac{k}{b} + \frac{1}{2b}\right) \right| < \epsilon + 3\delta.$$

In order to complete the proof in this case we simply choose $\delta > 0$ so that $\epsilon + 3\delta < \frac{1}{4}$.

V. GENERALIZATION TO n AND INFINITELY MANY INTERVALS

Repetition of the argument in Section IV allows us to successively split the interval $[0,1]$ in such a way that the corresponding Riesz basis for that interval also splits. For example, given the partition $0 = a_0 < a_1 < \cdots < a_n = 1$ of $[0,1]$, with Riesz basis $\mathcal{E}(\mathbb{Z})$, we can at the first stage split it into $[0,1] = [0, a_{n-1}] \cup [a_{n-1}, 1]$ thereby obtaining subsets Λ_n and Γ_n of $\mathbb{Z}$ such that $\mathcal{E}(\Lambda_n)$ is a Riesz basis for $L^2[a_{n-1}, 1]$ and such that $\mathcal{E}(\Gamma_n)$ is a Riesz basis for $L^2[0, a_{n-1}]$. At the next step, split $[0, a_{n-1}]$ into $[0, a_{n-2}] \cup [a_{n-2}, a_{n-1}]$ and correspondingly split Γ_n into Λ_{n-1} and Γ_{n-1} each with the appropriate Riesz basis property. Continuing in this fashion gives Theorem 1.1 where we at the end take $\Lambda_1 = \Gamma_1$. By choosing $\epsilon = \frac{1}{8(3(n-2)+1)}$ and $\delta = \epsilon$ at each step we guarantee that at each step the appropriate Avdonin bound is less than $\frac{1}{4}$.

Modifying the order in which the interval is split gives us Theorem 1.2 as follows. Given $[\alpha, \beta] \subseteq [0, 1]$ with partition $\alpha = a_0 < a_1 < \cdots < a_{n-1} < a_n = \beta$, we consider the corresponding partition of $[0, 1]$ given by

$$0 = b_0 < b_1 = \alpha < b_2 = a_1 < \cdots$$
$$< b_n = a_{n-1} < b_{n+1} = \beta < b_{n+2} = 1.$$

At the first step, split $[0, 1]$ into $[0, 1] = [0, \beta] \cup [\beta, 1]$, and $\mathbb{Z}$ into Γ_{n+1} and Λ_{n+1}, at the second step split $[0, \beta]$ into $[0, \beta] = [0, \alpha] \cup [\alpha, \beta]$, and Γ_{n+1} into Λ_0 and Γ_0. In subsequent steps, split $[\alpha, \beta]$ as desired obtaining sets $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ as required by the theorem.

There is no impediment to continuing to split $[0, 1]$ indefinitely. By choosing at the kth step $\epsilon_k = \delta_k$ so that $\epsilon_1 + 3\sum_{k=2}^{\infty} \epsilon_k < \frac{1}{4}$ we can guarantee that at each step the Avdonin bound is sufficiently small. Letting $\{a_k\}_{k \in \mathbb{Z}}$ be as in Theorem 1.3, we may at the first step split $[0, 1]$ into $[0, a_1] \cup [a_1, 1]$, and $\mathbb{Z}$ into $\Lambda_1 \cup \Gamma_1$, $[a_1, 1]$ into $[a_1, a_2] \cup [a_2, 1]$ and Γ_1 into $\Lambda_2 \cup \Gamma_2$, and so forth. By choosing $\epsilon_k = \delta_k$ so that $\epsilon_1 + 3\sum_{k=2}^{\infty} \epsilon_k < \frac{1}{4}$ we can guarantee that at the kth step the Avdonin bound is sufficiently small that Theorem 1.3(1) and (3) hold. Theorem 1.3(2) follows from Theorem 3.1.

REFERENCES

[1] S. Avdonin. On the question of Riesz bases of complex exponential function in l^2. *Vestnik Leningrad Univ. Ser. Mat.*, 13:5–12, 1974.
[2] Bernhard G. Bodmann, Peter G. Casazza, Vern I. Paulsen, and Darrin Speegle. Spanning and independence properties of frame partitions. *Proc. Amer. Math. Soc.*, 140(7):2193–2207, 2012.
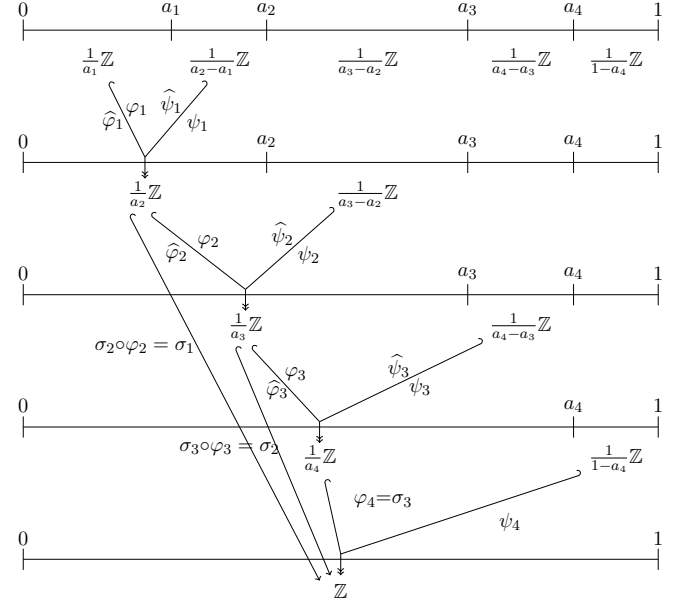
Fig. 1. Illustration of Theorem 1.1 with $n = 5$. In the initial step (fifth to fourth row) we use Theorem 3.2 to construct mappings $\varphi_4 : \frac{1}{a_4}\mathbb{Z} \to \mathbb{Z}$ and $\widehat{\psi}_k : \frac{1}{1-a_4}\mathbb{Z} \to \mathbb{Z}$ whose ranges partition $\mathbb{Z}$ into Γ_4 and Λ_4 and where $\mathcal{E}(\Lambda_4)$ and $\mathcal{E}(\Gamma_4)$ are Riesz bases for $L^2[a_4, 1]$ and $L^2[0, a_4]$ respectively. At the next step (fourth to third row) we construct mappings $\widehat{\varphi}_3 : \frac{1}{a_3}\mathbb{Z} \to \frac{1}{a_4}\mathbb{Z}$ and $\widehat{\psi}_3 : \frac{1}{a_4-a_3}\mathbb{Z} \to \frac{1}{a_4}\mathbb{Z}$ whose ranges partition $\frac{1}{a_4}\mathbb{Z}$. Taking $\sigma_3 = \varphi_4$ we can adjust $\widehat{\varphi}_3$ and $\widehat{\psi}_3$ to φ_3 and ψ_3 by Theorem 4.1 so that $\Gamma_3 = \text{Range}(\sigma_3 \circ \varphi_3)$ and $\Lambda_3 = \text{Range}(\sigma_3 \circ \psi_3)$ partition Γ_4 and where $\mathcal{E}(\Lambda_3)$ and $\mathcal{E}(\Gamma_3)$ are Riesz bases for $L^2[a_3, a_4]$ and $L^2[0, a_3]$ respectively. Continuing (third to second row) we partition $\frac{1}{a_3}\mathbb{Z}$ by means of mappings $\widehat{\varphi}_2$ and $\widehat{\psi}_2$ defined on $\frac{1}{a_2}\mathbb{Z}$ and $\frac{1}{a_3-a_2}\mathbb{Z}$ respectively from Theorem 3.2 then taking $\sigma_2 = \sigma_3 \circ \varphi_3$, defining φ_2 and ψ_2 via Theorem 4.1, we partition Γ_3 into $\Gamma_2 = \text{Range}(\sigma_2 \circ \varphi_2)$ and $\Lambda_3 = \text{Range}(\sigma_2 \circ \psi_2)$ so that $\mathcal{E}(\Lambda_2)$ and $\mathcal{E}(\Gamma_2)$ are Riesz bases for $L^2[a_2, a_3]$ and $L^2[0, a_2]$ respectively, and so on.

[3] Marcin Bownik, Peter G. Casazza, Adam W. Marcus, and Darrin Speegle. Improved bounds in Weaver and Feichtinger conjectures. *Journal fr die reine und angewandte Mathematik (Crelles Journal)*, 0(0), 2016.
[4] M. I. Kadeć. The exact value of the PaleyWiener constant. *Dokl. Akad. Nauk. SSSR 155*, 155:1253–1254, 1964. (English translation Sov. Math. Dokl. 5(2), 559-561, 1964).
[5] Gady Kozma and Shahaf Nitzan. Combining Riesz bases. *Invent. Math.*, 199(1):267–285, 2015.
[6] Gady Kozma and Shahaf Nitzan. Combining Riesz bases in $\mathbb{R}^d$. *Rev. Mat. Iberoam.*, 32(4):1393–1406, 2016.
[7] Basarab Matei and Yves Meyer. A variant of compressed sensing. *Rev. Mat. Iberoam.*, 25(2):669–692, 2009.
[8] Götz Pfander, Shauna Revay, and David Walnut. Exponential bases for partitions of intervals. Preprint, 2019.
[9] Shauna Revay. *Unions of Riesz bases of exponentials for bandlimited signals*. PhD thesis, George Mason University, Department of Mathematical Sciences, Fairfax, VA 22030, May 2018.
[10] Kristian Seip. A simple construction of exponential bases in L^2 of the union of several intervals. *Proc. Edinburgh Math. Soc. (2)*, 38(1):171–177, 1995.