Dynamical Sampling with a Burst-like Forcing Term

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***Abstract*—In this paper we consider the problem of recovery of a burst-like forcing term in the framework of dynamical sampling. We introduce the notion of a sensing limit of a collection of samples with respect to a semigroup and indicate its fundamental role in the solvability of the problem.**

I. Introduction

In this paper, we consider the problem of recovering an unknown source term $f = f(x,t)$ from space-time samples $\{u(x_i,t_k)\}$ of a function $u = u(x,t)$ that evolves in time due to the action of a known evolution operator A and the forcing function f. The variable $x \in \mathbb{R}^d$ is the "spatial" variable, while $t \in \mathbb{R}_+$ represents time. For each fixed t, $u(\cdot,t)$ can be viewed as a vector $u(t)$ in a Hilbert space $\mathcal{H}$. With this identification we get the following abstract initial value problem:

$$\begin{cases} \dot{u}(t) = Au(t) + f(t) \\ u(0) = 0, \end{cases} \tag{1}$$

Above $\dot{u} : \mathbb{R}_+ \to \mathcal{H}$ is the time derivative of u, $f : \mathbb{R}_+ \to \mathcal{H}$, and $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$ is a generator of a strongly continuous semigroup T. Recall that a strongly continuous semigroup is a map $T : \mathbb{R}_+ \to B(\mathcal{H})$ (where $B(\mathcal{H})$ is the space of all bounded linear operators on $\mathcal{H}$), which satisfies

1) $T(0) = I$,
2) $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$, and
3) $\|T(t)h - h\| \to 0$ as $t \to 0$ for all $h \in \mathcal{H}$.

The operator A is said to be a generator of the semigroup T if, given

$$D(A) = \{h \in \mathcal{H} : \lim_{t\to 0^+} \frac{1}{t}(T(t)h - h) \text{ exists}\},$$

then A satisfies

$$Ah = \lim_{t\to 0^+} \frac{1}{t}(T(t)h - h), \ h \in D(A).$$

The semigroup T is said to be uniformly continuous if $\|T(t) - I\| \to 0$ as $t \to 0$. In this case, $D(A) = \mathcal{H}$.

A prototypical example is $A = \Delta$ – the Laplacian operator on Euclidean space. For this case, f represents the unknown "heat source" that we seek to recover from the space-time samples of the temperature u. In this paper, we only consider "burst-like" forcing terms of the form

$$f(t) = \sum_{i=1}^{N} f_i \delta(t - t_i), \tag{2}$$

for some unknown $N \in \mathbb{N}$, with $0 \leq t_1 < t_2 \ldots < t_N < t_{N+1} = L$ and $f_i \in V$, where V is a subspace of $\mathcal{H}$ and δ is the Dirac delta-function.

This work was motivated by the problem of isolating localized source terms considered in [6], [7].

II. Problem statement

With the notation given in the introduction, the problem studied in this paper can be stated as follows.

Problem 2.1: Find the conditions on a semigroup T, a countable set $\mathcal{G} \subset \mathcal{H}$, and a number $L \in \mathbb{R}_+$ that allow one to (stably) recover any f of the form (2) from the set of samples

$$\{\langle u(t), g\rangle : g \in \mathcal{G}, t \in [0, L]\}. \tag{3}$$

Since f is of the form (2), and the solution of (1) can be represented [5, p. 436] as

$$u(t) = \int_0^t T(t-\tau) f(\tau) d\tau, \tag{4}$$

the samples in (3) become

$$\langle u(t), g\rangle = \sum_{t_i \le t} \langle T(t-t_i) f_i, g\rangle, g \in \mathcal{G}.$$

In our recent papers (see, e.g., [1], [2], [3], [4] and the references therein) we considered the problem of recovering a vector $f \in \mathcal{H}$ from the samples $\{\langle A^n f, g\rangle : g \in \mathcal{G}, n = 0, 1, \ldots, L\}$. We wish to utilize some of those ideas for Problem 2.1, thus putting it into the general framework of dynamical sampling.

III. Results

The difficulty of solving Problem 2.1 can be illustrated as follows.

Example 3.1: It may happen, that one cannot determine a source term of the form (2) uniquely. Let T be the semigroup of translations acting on $L^2(\mathbb{R})$, i.e. $[T(t)f](s) = f(s-t)$. Let $h_1 = \chi_{[0,1]}, h_2 = \chi_{[2,3]}$ and $V = \text{span}\{h_1, h_2\}$. Let $g = \chi_{[1,2]} + 2\chi_{[3,4]}$ be a single measurement function. We see that distinct elements of V, $f = (h_1 - 1/2h_2)\delta(t)$ and $\tilde{f} = h_2\delta(t-2)$, result in the same samples over any interval.

In order to understand this problem, we introduce the notion of a *sensing limit* of a system $\mathcal{G}$ with respect to the semigroup T.

Given a subspace V of a Hilbert space $\mathcal{H}$, let P_V be the orthogonal projection onto V. From the theory of frames, we have that the system $\{P_V T^*(t) g : g \in \mathcal{G},\ t \in [0, \alpha]\}$ is a (semicontinuous) frame for V if there exist $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{g \in \mathcal{G}} \int_0^\alpha |\langle f, P_V T^*(t) g\rangle|^2 dt \le B\|f\|^2$$

for all $f \in V$. We can thereby define the *sensing limit* ℓ as follows:

$$\ell = \ell_V(\mathcal{G}, T) = \inf\{\alpha > 0 :$$
$$\{P_V T^*(t) g : g \in \mathcal{G}, t \in [0, \alpha]\} \text{ is a frame for } V\}.$$

We call the system $\mathcal{G}$ *immediately sensing* if $\ell = 0$.

One can verify that in Example 3.1 we have $\ell_V(\mathcal{G}, T) = 2$.

Our first result for the recovery of f can now be stated as follows.

Theorem 3.1: Assume that the numbers $\{t_i\}_{i=1}^N \subset [0, L)$ with $0 = t_1 < t_2 < \ldots < t_N < t_{N+1} = L$ are known, and the sensing limit of $\mathcal{G}$ with respect to T satisfies

$$\ell_V(\mathcal{G}, T) < \min\{t_{i+1} - t_i : i = 1, \ldots, N\}.$$

Then f in (2) can be stably recovered from (3).

In the case when the values of t_i are unknown, the problem can be solved under more stringent conditions on ℓ.

Theorem 3.2: Assume that $\ell_V(\mathcal{G}, T) = 0$. Then $N \in \mathbb{N}$, $\{t_i\}_{i=1}^N \subset [0, L)$ with $0 = t_0 < t_1 < \ldots < t_N < L$ and f in (2) can all be stably recovered from (3).

Theorems 3.1 and 3.2 motivated us to study how various conditions on T and $\mathcal{G}$ affect ℓ. For example, we have the following result.

Theorem 3.3: Assume that T is a uniformly continuous semigroup and V is a finite dimensional subspace of $\mathcal{H}$. Then $\ell_V(\mathcal{G}, T) \in \{0, \infty\}$.

As an immediate consequence of Theorems 3.3 and 3.2, we get

Corollary 3.1: Let the following three assumptions hold.

1. $\{t_i\}_{i=1}^N \subset [0, L)$ with $0 = t_1 < t_2 < \ldots < t_N < t_{N+1} = L$ and $\{t_i\}_{i=1}^N$ are unknown.
2. The sensing limit is finite, i.e., there exists a positive number α such that $\{P_V T^*(t) g : g \in \mathcal{G},\ t \in [0, \alpha]\}$ is a frame for V.
3. T is a uniformly continuous semigroup and V is a finite dimensional subspace of $\mathcal{H}$.

Then f in (2) can be stably recovered from (3).

IV. Concluding Remarks

Our results suggest that many of the questions that are relevant for the dynamical sampling problem of burst-like sources can be answered once the behavior of the sensing limit is well understood. For example, finding weak sufficient conditions under which $\ell_V(\mathcal{G}, T) = 0$ or is finite would be an important step.

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