A Trace Result of Shearlet Coorbit Spaces on Lines

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***Abstract*—We study traces of certain subspaces of shearlet coorbit spaces on lines in $\mathbb{R}^2$ which extends the results for horizontal and vertical lines from [2].**

I. Introduction

In recent years shearlets have been shown to be efficient extractors of the directional information present in a signal. This makes them well-suited for various applications like [11], [13]. Moreover, quite surprisingly, the shearlet transform has the outstanding property to stem from a square integrable group representation [1]. This remarkable fact provides the opportunity to design associated canonical smoothness spaces, so-called shearlet coorbit spaces [3], [5] by applying the general coorbit theory derived by Feichtinger and Gröchenig [8], [9]. To understand the structure of shearlet coorbit spaces and in view of possible applications it is desirable to know how these new spaces behave under trace operations. In [4], embedding theorems into Besov spaces were employed to establish embedding relations of traces of certain subspaces of shearlet coorbit spaces on the boundary of smooth domains in $\mathbb{R}^d$ with $d = 2, 3$.

In this paper, we follow another approach. We directly extend the trace results for hyperplanes from [2] to arbitrary lines in $\mathbb{R}^2$ using the characterization of homogeneous Besov spaces by K-atoms.

II. Preliminary

In this section, we recall basic facts from shearlet coorbit theory [2], [5]. For $a \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $s \in \mathbb{R}$, let

$$A_a := \begin{bmatrix} a & 0 \\ 0 & \operatorname{sgn}(a)\sqrt{|a|} \end{bmatrix} \quad \text{and} \quad S_s := \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix},$$

be the *parabolic scaling matrix* and the *shear matrix*. The *shearlet group* $\mathbb{S}$ is defined to be the set $\mathbb{R}^* \times \mathbb{R} \times \mathbb{R}^2$ endowed with the group operation

$$(a, s, t)(a', s', t') = (aa', s + s'\sqrt{|a|}, t + S_s A_a t').$$

Its left-invariant Haar measure is given by $\mu_{\mathbb{S}} = |a|^{-3}\, da\, ds\, dt$. The mapping π is defined by $\pi(a, s, t)\, \psi(x) := |a|^{-\frac{3}{4}} \psi(A_a^{-1} S_s^{-1}(x - t))$ is a unitary representation of $\mathbb{S}$. It is also *square integrable*, i.e., irreducible and there exists a nontrivial *shearlet* $\psi \in L_2(\mathbb{R}^2)$ fulfilling the *admissibility condition*

$$\int_{\mathbb{S}} |\langle f, \pi(a, s, t)\psi\rangle|^2\, d\mu_{\mathbb{S}}(a, s, t) < \infty.$$

For a shearlet ψ the transform $\mathcal{SH}_\psi : L_2(\mathbb{R}^2) \to L_2(\mathbb{S})$ defined by

$$\mathcal{SH}_\psi(f)(a, s, t) := \langle f, \pi(a, s, t)\psi\rangle,$$

is called *continuous shearlet transform*.

Let w be a real-valued, positive, continuous and submultiplicative weight on $\mathbb{S}$ which fulfills in addition all the coorbit-theory conditions as stated in [12, Section 2.2]. Consider $L_{p,w}(\mathbb{S}) := \{F \text{ measurable} : Fw \in L_p(\mathbb{S})\}$, $p \in [1, \infty]$ with the norm $\|F\|_{L_{p,w}} := \left(\int_{\mathbb{S}} |F(g)w(g)|^p\, d\mu_{\mathbb{S}}(g)\right)^{1/p}$. For a vector $\psi (\neq 0)$ contained in $\{\psi \in L_2(\mathbb{R}^2) : \mathcal{SH}_\psi(\psi) = \langle \psi, \pi(\cdot)\psi\rangle \in L_{1,w}(\mathbb{S})\}$, we introduce the space $\mathcal{H}_{1,w} := \{f \in L_2(\mathbb{R}^2) : \mathcal{SH}_\psi(f) = \langle f, \pi(\cdot)\psi\rangle \in L_{1,w}(\mathbb{S})\}$, with norm $\|f\|_{\mathcal{H}_{1,w}} := \|\mathcal{SH}_\psi(f)\|_{L_{1,w}(\mathbb{S})}$ and its anti-dual $\mathcal{H}_{1,w}^\sim$, the space of all continuous conjugate-linear functionals on

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$\mathcal{H}_{1,w}$. The following sesquilinear form on $\mathcal{H}_{1,w}^\sim \times \mathcal{H}_{1,w}$ is well-defined:

$$\mathcal{SH}_\psi(f)(a,s,t) := \langle f, \pi(a,s,t)\psi\rangle_{\mathcal{H}_{1,w}^\sim \times \mathcal{H}_{1,w}}.$$

We are interested in the special weights

$$m(a,s,t) = m(a) := |a|^{-r}, \quad r \geq 0,$$

where m be a w-moderate weight on $\mathbb{S}$ which means $m(xyz) \leq w(x)m(y)w(z)$ for all $x,y,z \in \mathbb{S}$ and use the abbreviation $L_{p,r}(\mathbb{S}) := L_{p,m}(\mathbb{S})$. The Banach spaces $\mathcal{SC}_{p,r} := \{f \in \mathcal{H}_{1,w}^\sim : \mathcal{SH}_\psi(f) \in L_{p,r}(\mathbb{S})\}$, $\|f\|_{\mathcal{SC}_{p,r}} := \|\mathcal{SH}_\psi(f)\|_{L_{p,r}(\mathbb{S})}$ are called *shearlet coorbit spaces*. A (countable) family $X = \{g_i := (a_i,s_i,t_i) : i \in \mathcal{I}\}$ in $\mathbb{S}$ is said to be *U-dense* if $\bigcup_{i\in\mathcal{I}} g_i U = \mathbb{S}$, and *separated* if for some compact neighborhood Q of $e = (1,0,0) \in \mathbb{R}^* \times \mathbb{R} \times \mathbb{R}^2$ we have $g_iQ \cap g_jQ = \emptyset, i \neq j$, and *relatively separated* if X is a finite union of separated sets. Based on U-dense and relatively separated sets the existence of atomic decompositions of shearlet coorbit spaces is ensured [3], [7]. It was shown in [2] that for a neighborhood of the identity $U \supset [\alpha^{-\frac{1}{2}}, \alpha^{\frac{1}{2}}) \times [-\frac{\beta}{2}, \frac{\beta}{2}) \times [-\frac{\tau}{2}, \frac{\tau}{2})^2$, $\alpha > 1,\ \beta, \tau > 0$, the set $X := \Big\{(\varepsilon\alpha^{-j}, \beta\alpha^{-\frac{j}{2}}k, S_{\beta\alpha^{-\frac{j}{2}}k} A_{\alpha^{-j}}\tau l) : j \in \mathbb{Z}, k \in \mathbb{Z}, l \in \mathbb{Z}^2, \varepsilon \in \{-1,1\}\Big\}$ is U-dense and relatively separated. Without loss of generality, we can restrict our attention to the $\varepsilon = +1$ and $\beta = \tau = 1$. For $a := \alpha^{-j}$, $s := \alpha^{-\frac{j}{2}}k$ and $t := S_{\alpha^{-\frac{j}{2}}k} A_{\alpha^{-j}} l$ we use the abbreviation $\psi_{j,k,l} := \pi(a,s,t)\psi$.

Every function $f \in \mathcal{SC}_{p,r}$ can be written as

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}^2} c(j,k,l)\psi_{j,k,l}(x) = \sum_{j,k,l} c(j,k,l)\alpha^{\frac{3}{4}j}\psi(\alpha^j x_1 - \alpha^{\frac{j}{2}}kx_2 - l_1, \alpha^{j/2}x_2 - l_2).$$

For fixed $\psi \in B_w$ as defined in [3], we denote by $\mathcal{SCC}_{p,r}$ the closed subspace of $\mathcal{SC}_{p,r}$ consisting of functions related to cone-adapted shearlets:

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{|k|\leq\alpha^{\frac{j}{2}}} \sum_{l\in\mathbb{Z}^2} c(j,k,l)\psi_{j,k,l}(x),$$

i.e. if $T : \mathcal{SC}_{p,r} \to \ell_{p,m}$ is the linear, continuous operator mapping $f \mapsto \{c_{j,k,l}\}$, then $\mathcal{SCC}_{p,r}$ is the pre-image of T of the closed set $\{c_{j,k,l} : j \in \mathbb{Z}, |k| \leq \alpha^{\frac{j}{2}}, l \in \mathbb{Z}^2\}$.

Finally, we need the characterization of homogeneous Besov spaces $B^\sigma_{p,q}$ from [10]. For $\alpha > 1$, $D > 1$ and $K \in \mathbb{N}_0$, a K times differentiable function a on $\mathbb{R}^d$ is called a *K-atom* if the following two conditions are fulfilled:

A1) $\operatorname{supp} a \subset DQ_{j,m}(\mathbb{R}^d)$ for some $m \in \mathbb{R}^d$, where $DQ_{j,m}(\mathbb{R}^d)$ denotes the cube in $\mathbb{R}^d$ centered at $\alpha^{-j}m$ with sides parallel to the coordinate axes and side length $2D\alpha^{-j}$.

A2) $|D^\gamma a(x)| \leq c\alpha^{|\gamma|j}$ for $|\gamma| \leq K$.

Now the homogeneous Besov spaces can be characterized as follows.

Theorem 2.1: Let $D > 1$ and $K \in \mathbb{N}_0$ with $K \geq 1 + \lfloor\sigma\rfloor$ be fixed. Let $1 \leq p \leq \infty$. Then $f \in B^\sigma_{p,q}$ if and only if it can be represented [1] as

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{l\in\mathbb{Z}^d} \lambda(j,l)a_{j,l}(x), \tag{1}$$

where the $a_{j,l}$ are K-atoms with $\operatorname{supp} a_{j,l} \subset DQ_{j,l}(\mathbb{R}^d)$. And

$$\|f\|_{B^\sigma_{p,q}} \sim \inf\Big(\sum_{j\in\mathbb{Z}} \alpha^{j(\sigma-\frac{d}{p})q}\big(\sum_{l\in\mathbb{Z}^d} |\lambda(j,l)|^p\big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}$$

where the infimum is taken over all admissible representations (1).

III. Traces on lines

In this section we prove a trace theorem on general lines. Let $L(x) := (x, \ell(x))$, $x \in \mathbb{R}$ be a line in $\mathbb{R}^2$ not coinciding with the horizontal and vertical axes, where mapping $\ell(x)$ is defined by $\ell(x) = ax$, where $a \in \mathbb{R}_{>0}$. We define Besov spaces on the line $B^{\sigma_1}_{p,p}(L)$ as

$$f \in B^{\sigma_1}_{p,p}(L) \quad \text{if and only if} \quad f(x,\ell(x)) \in B^{\sigma_1}_{p,p}(\mathbb{R}^2). \tag{2}$$

[1] In the sense of distributions, a-posteriori implying norm convergence for $p < \infty$.

Theorem 3.1: Let $\ell(x) := ax$ with slope $a \in \mathbb{R}_{>0}$ and let $\mathrm{Tr}_L f$ denote the restriction of f to L, i.e., $\mathrm{Tr}_L f(x) := f(x, ax)$. Then $\mathrm{Tr}_L(\mathcal{SCC}_{p,r}) \subset B^{\sigma_1}_{p,p}(L) + B^{\sigma_2}_{p,p}(L)$ holds true, where σ_i, $i = 1, 2$, have to fulfill

$$\sigma_1 + \lfloor \sigma_1 \rfloor \leq 2r - \frac{9}{2} + \frac{3}{p}, \quad \text{and} \quad \sigma_2 \leq 2r - \frac{5}{2} + \frac{2}{p}.$$

Proof: The proof is performed by first of all considering a subspace of the Schwartz functions which is dense in the shearlet coorbit spaces, see [6] for details. In this space, point evaluations are clearly well-defined. Then the result follows by continuous extension arguments. We can write $f \in \mathcal{SCC}_{p,r}$ in the form

$$f(x,y) = \sum_{j\in\mathbb{Z}} \sum_{|k|\leq\alpha^{j/2}} \sum_{l\in\mathbb{Z}^2} c(j,k,l)\alpha^{\frac{3}{4}j} \psi(\alpha^j x - \alpha^{j/2}ky - l_1, \alpha^{j/2}y - l_2).$$

We split f as $f_1 + f_2$, where

$$f_1(x,y) = \sum_{j\geq 0} \sum_{|k|\leq\alpha^{j/2}} \sum_{l\in\mathbb{Z}^2} c(j,k,l)\alpha^{\frac{3}{4}j} \psi(\alpha^j x - \alpha^{j/2}ky - l_1, \alpha^{j/2}y - l_2),$$

$$f_2(x,y) = \sum_{j<0} \sum_{l\in\mathbb{Z}^2} c(j,0,l)\alpha^{\frac{3}{4}j} \psi(\alpha^j x - l_1, \alpha^{j/2}y - l_2).$$

By [6, Theorem 3.2], we can choose ψ compactly supported in $[-D, D] \times [-D, D]$ for some $D > 1$. Further, ψ can be normalized such that the partial derivatives (γ_1, γ_2) of order $\gamma = \gamma_1 + \gamma_2$ with $0 \leq \gamma \leq K := \max\{K_1, K_2\}$ are not larger than 1, where $K_1 := 1 + \lfloor\sigma_1\rfloor$, $K_2 := 1 + \lfloor\sigma_2\rfloor$. Then we obtain

$$\mathrm{Tr}_L f(x) = f(x, ax) = \sum_{j\in\mathbb{Z}} \sum_{|k|\leq\alpha^{j/2}} \sum_{l\in\mathbb{Z}^2} c(j,k,l)\alpha^{\frac{3}{4}j} \psi(\alpha^j x - \alpha^{j/2}kax - l_1, \alpha^{j/2}ax - l_2).$$

The support assumptions on ψ and $a > 0$ shows that if $0 \neq \psi(\alpha^j x - \alpha^{j/2}kax - l_1, \alpha^{j/2}ax - l_2)$, then

$$\alpha^{-j/2}a^{-1}(l_2 - D) \leq x \leq \alpha^{-j/2}a^{-1}(l_2 + D)$$

and

$$(\alpha^{j/2}a^{-1} - k)l_2 - (\alpha^{j/2}a^{-1} + |k| + 1)D \leq l_1 \leq (\alpha^{j/2}a^{-1} - k)l_2 + (\alpha^{j/2}a^{-1} + |k| + 1)D.$$

Let $I_{k,l_2,j} := \{r \in \mathbb{Z} : |r - (\alpha^{j/2}a^{-1} - k)l_2| \leq D(1 + |k| + \alpha^{j/2}a^{-1})\}$ and $J_{l_2,j} := \{r \in \mathbb{Z} : |r - \alpha^{j/2}a^{-1}l_2| \leq D(1 + \alpha^{j/2}a^{-1})\}$. Now we can write $\mathrm{Tr}_L f$ as

$$\begin{aligned}\mathrm{Tr}_L f(x) &= \sum_{j\geq 0}\sum_{l_2\in\mathbb{Z}} \lambda(j,l_2)a_{j,l_2}(x) + \sum_{j<0}\sum_{l_2\in\mathbb{Z}} \lambda(j,l_2)a_{j,l_2}(x) \\ &= \mathrm{Tr}_L f_1(x) + \mathrm{Tr}_L f_2(x),\end{aligned}$$

where for $j \geq 0$,

$$\lambda(j,l_2) := \alpha^{\frac{3}{4}j}\alpha^{\frac{j}{2}K_1} \sum_{|k|\leq\alpha^{j/2}} \sum_{l_1\in I_{k,l_2,j}} |\, c(j,k,l_1,l_2) \,|$$

we set $a_{j,l_2}(x) := 0$ if $\lambda(j,l_2) = 0$ and otherwise

$$a_{j,l_2}(x) := \lambda(j,l_2)^{-1}\alpha^{\frac{3}{4}j} \sum_{|k|\leq\alpha^{j/2}} \sum_{l_1\in I_{k,l_2,j}} c(j,k,l_1,l_2) \psi(\alpha^j x - \alpha^{j/2}kax - l_1, \alpha^{j/2}ax - l_2).$$

For $j < 0$ we use

$$\lambda(j,l_2) := \alpha^{\frac{3}{4}j} \sum_{l_1\in J_{l_2,j}} |\, c(j,0,l_1,l_2) \,|$$

and we set $a_{j,l_2}(x) := 0$ if $\lambda(j,l_2) = 0$ and otherwise

$$a_{j,l_2}(x) := \lambda(j,l_2)^{-1}\alpha^{\frac{3}{4}j} \sum_{l_1\in J_{l_2,j}} c(j,0,l_1,l_2) \psi(\alpha^j x - l_1, \alpha^{j/2}ax - l_2).$$

Since the support of ψ is in $[-D, D] \times [-D, D]$, we have for $h(x) := \psi(\alpha^j x - \alpha^{j/2}kax - l_1, \alpha^{j/2}ax - l_2)$ that $\operatorname{supp} h \subseteq [\alpha^{-j/2}a^{-1}l_2 - \alpha^{-j/2}a^{-1}D, \alpha^{-j/2}a^{-1}l_2 + \alpha^{-j/2}a^{-1}D]$ and therefore $\operatorname{supp} a_{j,l_2} \subseteq [\alpha^{-j/2}a^{-1}l_2 - \alpha^{-j/2}a^{-1}D,$

$\alpha^{-j/2}a^{-1}l_2+\alpha^{-j/2}a^{-1}D]$. Note that we have only involved the support condition for the second component of ψ; but those for the first component does not further reduce the size of the possible support of a_{j,l_2}. We show that the a_{j,l_2} are atoms with respect to the scaling with $\alpha^{1/2}$. For $j \geq 0$ and $|k| \leq \alpha^{j/2}$ we have $|D^\gamma h(x)| \leq (2a+1)^{\gamma/2}\alpha^{j\gamma}$. Hence, for $\gamma \leq K_1$, we get $\alpha^{-\frac{j}{2}K_1}|D^\gamma h(x)| \leq c\alpha^{\frac{j}{2}\gamma}$, $c := (2a+1)^{\frac{K_1}{2}}$ and then $|D^\gamma a_{j,l_2}(x)| \leq c\alpha^{\frac{j}{2}\gamma}$. Thus a_{j,l_2} are K_1 atoms on L. We have

$$\| \operatorname{Tr}_L f_1\|_{B^{\sigma_1}_{p,p}} \leq \Big(\sum_{j\geq 0} \alpha^{\frac{j}{2}(\sigma_1-\frac{1}{p}+\frac{3}{2}+K_1)p} \sum_{l_2\in\mathbb{Z}} \big(\sum_{|k|\leq \alpha^{j/2}} \sum_{l_1\in I_{k,l_2,j}} |\ c(j,k,l_1,l_2)\ |\ \big)^p \Big)^{\frac{1}{p}}.$$

Now we use the inequality $\big(\sum_{i=1}^N |z_i|\big)^p \leq N^{p-1}\sum_{i=1}^N |z_i|^p$ together with the fact that $I_{k,l_2,j}$ contains atmost $c\alpha^{j/2}$ elements and that there are atmost $c\alpha^{j/2}$ integers with absolute value smaller or equal than $\alpha^{j/2}$. We obtain

$$\begin{aligned}
\| \operatorname{Tr}_L f_1\|_{B^{\sigma_1}_{p,p}} &\lesssim \Big(\sum_{j\geq 0} \alpha^{\frac{j}{2}(\sigma_1-\frac{1}{p}+\frac{3}{2}+K_1)p}\alpha^{j(p-1)} \sum_{|k|\leq \alpha^{j/2}} \sum_{l\in\mathbb{Z}^2} |\ c(j,k,l)\ |^p \Big)^{\frac{1}{p}} \\
&\lesssim \big(\sum_{j\in\mathbb{Z}} \alpha^{jpr} \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^2} |\ c(j,k,l)\ |^p \big)^{\frac{1}{p}} \\
&\lesssim \|f\|_{\mathcal{SC}_{p,r}}.
\end{aligned}$$

with $r \geq \frac{1}{2}(\sigma_1 + \lfloor \sigma_1 \rfloor + \frac{9}{2} - \frac{3}{p})$.

For $j < 0$ we have $\alpha^j \leq \alpha^{j/2}$ such that $|D^\gamma h(x)| \leq c\alpha^{\frac{j}{2}\gamma}$ which holds also true for $D^\gamma a_{j,l_2}$. Therefore a_{j,l_2} are K atoms on L. Now we obtain similarly as above by regarding that $J_{l_2,j}$ has $c\alpha^{j/2}$ elements

$$\begin{aligned}
\| \operatorname{Tr}_L f_2\|_{B^{\sigma_2}_{p,p}} &\leq \big(\sum_{j\in\mathbb{Z}} \alpha^{\frac{j}{2}(\sigma_2-\frac{1}{p})p} \sum_{l_2\in\mathbb{Z}} |\ \lambda(j,l_2)\ |^p \big)^{\frac{1}{p}} \\
&\lesssim \big(\sum_{j<0} \alpha^{\frac{j}{2}(\sigma_2-\frac{1}{p}+\frac{3}{2})p}\alpha^{\frac{j}{2}(p-1)} \sum_{l\in\mathbb{Z}^2} |\ c(j,0,l)\ |^p \big)^{\frac{1}{p}} \\
&\lesssim \big(\sum_{j\in\mathbb{Z}} \alpha^{jpr} \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^2} |\ c(j,k,l)\ |^p \big)^{\frac{1}{p}} \\
&\lesssim \|f\|_{\mathcal{SC}_{p,r}}
\end{aligned}$$

with $r \geq \frac{1}{2}(\sigma_2 + \frac{5}{2} - \frac{2}{p})$. This finishes the proof.

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